On the symmetries of the one-dimensional Hénon-Lane-Emden systems

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Abstract: In this talk we show some recent results regarding Lane-Emden systems with one independent variable. In particular, we extend the previous results obtained in [Y. Bozhkov and I. L. Freire, On the Lane Emden system in dimension one, Appl. Math. Comp., 218 (2012) 10762-10766].

1 Introduction

In [5] Hénon proposed a new model for polytropes, which was called by him as generalised polytropes. He considered in the mentioned reference models in which the distribution function not only depends on the energy per unit of mass of a star, but also depends on the angular momentum. With this hypothesis, the radial form of the Poisson equation derived by him (up to notation) was

\[ u''(x) + \frac{2}{x} u'(x) + x^n u^p = 0, \] (1)

The dependence on the angular momentum assumed by Hénon gives rise to the explicit dependence on the radial variable \( x \). Whenever \( m = 0 \) equation (1) is reduced to the well-known Lane-Emden equation, which was also employed to modelling polytropic stars. Therefore this equation is a generalisation of the Lane-Emden equation.

Equation (1) can easily be generalised to the following system

\[
\begin{cases}
  u''(x) + \frac{n-1}{x} u'(x) + x^\alpha v(x)^q = 0, \\
  v''(x) + \frac{n-1}{x} v'(x) + x^\beta u(x)^p = 0.
\end{cases}
\] (2)

System (2) can arise as a stationary case of certain reaction-diffusion systems, see [6]. Mathematically, whenever \( \alpha = \beta = 0 \), system (2) is reduced to the well-known Lane-Emden system

\[
\begin{cases}
  u''(x) + \frac{n-1}{x} u'(x) + v(x)^q = 0, \\
  v''(x) + \frac{n-1}{x} v'(x) + u(x)^p = 0,
\end{cases}
\] (3)

which has been studied since long time ago from different approaches. Particularly, since Bozhkov and Gilli Martins’ seminal paper [1] on symmetry analysis of system (3), a considerable number of works have been done considering this type of systems. More recently, the first author jointly with Y. Bozhkov have also considered a bidimensional Lane-Emden system [2].
In [3] it was carried out a complete Lie group classification of the system

\[
\begin{align*}
    u''(x) + v(x)^q &= 0, \\
    v''(x) + u(x)^p &= 0
\end{align*}
\] (4)

under the assumption \( p, q \notin \{0, 1\} \). This restriction was made in order to deal only with nonlinear cases.

Inspired on the recent work [3], the first author proposed to the second the problem of Lie group classification of the following system

\[
\begin{align*}
    u''(x) + x^\alpha v(x)^q &= 0, \\
    v''(x) + x^\beta u(x)^p &= 0
\end{align*}
\] (5)

for his master degree dissertation.

In particular, from the Lie group classification, it is obtained that the dilational generator

\[
X_{p,q,\alpha,\beta} = (1 - pq)x \frac{\partial}{\partial x} + [(\alpha + q\beta) + 2(1 + q)]u \frac{\partial}{\partial u} + [(\beta + p\alpha) + 2(1 + p)]v \frac{\partial}{\partial v}
\] (6)

is a Lie point symmetry of the system (5) for any values of \( \alpha, \beta, p \) and \( q \). However, we also prove the following result

**Theorem.** The Lie point symmetry generator (6) of the Lane-Emden system with Hénon-type weights is a variational symmetry if and only if \( \alpha, \beta, p \) and \( q \) satisfy the equation

\[
3 + \alpha + \beta + pq + 2(p + q) + pq = 0.
\] (7)

Whenever \( p \neq -1 \) and \( q \neq -1 \), equation (7) is equivalent to

\[
\frac{1 + \beta}{p + 1} + \frac{1 + \alpha}{q + 1} = -1,
\] (8)

which generalises the previous results obtained in [3], that is, a dilation is a Noether symmetry of the system (4) if and only if its parameters belong to the critical hyperbola (8). Setting \( \alpha = \beta = 0 \) into (8) we recover the same results as in [3].

We observe that for \( n \geq 3 \), the critical hyperbola is (see [4])

\[
\frac{n + \beta}{p + 1} + \frac{n + \alpha}{q + 1} = n - 2.
\] (9)

In this talk we will show the Lie point symmetries, Noether symmetries and first integrals to the considered equation. It will also be presented a three-parametric family of solutions to the case \( \alpha = \beta = 0 \) and \( p = q = -3 \).

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**Referências**


