Exploring Monte Carlo Method to Access the Dynamical Behavior of a Continuous Random System

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Abstract: The dynamics of a mechanical system depends on some parameters such as physical and geometrical properties, external and internal loading, initial and boundary conditions, etc. If all of these parameters are constant, then the system is deterministic and its behavior is described by a single set of differential equations. On the other hand, if one or more of the these parameters are random, the mechanical system is stochastic and there is a family of differential equations sets (one for each realization of the random system) associated to the same. Therefore, it is necessary to compute statistics of the random system realizations to characterize the dynamics of the stochastic mechanical system. This work illustrates the analysis of a one-dimensional elastic bar, with random elastic modulus and subjected nonlinear external force. The analysis is done through Monte Carlo method, where a large amount of realizations of the random system is obtained via numerical simulations. Then statistics of these realizations are computed, and these statistics reveal information about the average behavior of the random system.

1 Introduction

The dynamics of a mechanical system depends on some parameters such as physical and geometrical properties, external and internal loading, initial and boundary conditions, constraints. Most of the theoretical models used to study a mechanical system assume nominal values for these parameters, such that the model gives one response for a given particular input. In this case the system is deterministic and its behavior is described by a single set of differential equations. However, in the real systems, the system parameters do not have a fixed value since these systems are subjected to measurement uncertainties, imperfections in manufacturing processes, change of properties. This variability in the set of system parameters leads to a large number of possible system responses, for a given particular input. Now the system is stochastic and there is a family of differential-equations sets (one for each realization of the random system) associated to the same.

This work aims to illustrate the use of Monte Carlo simulation [3] to compute uncertainty propagation in dynamics of a continuous random system. For this purpose it presents the analysis of a bar with random elastic modulus and prescribed boundary conditions, say, fixed at one end and attached to a punctual mass and two springs (one linear and another nonlinear) on the other extreme.
2 Mathematical Modeling

2.1 Deterministic Approach

The continuous system of interest is the one-dimensional elastic bar shown in Figure 1.

\begin{align*}
\rho A \frac{\partial^2 u}{\partial t^2}(x,t) + c \frac{\partial u}{\partial t}(x,t) &= \frac{\partial}{\partial x} \left( EA \frac{\partial u}{\partial x}(x,t) \right) + f(x,t), \\
\end{align*}

which is valid for $0 < x < L$ and $0 < t < T$, being $L$ the bar unstretched length and $T$ a finite instant of time. In this equation $\rho$ is the mass density, $E$ is the elastic modulus, $A$ is the circular cross section area, $c$ is the damping coefficient, and

$$f(x,t) = \sigma \phi_1(x) \sin(\nu_1 t),$$

is an external force depending on position $x$ and time $t$, where $\sigma$ is the force amplitude, $\phi_1(x)$ and $\nu_1$ are the first natural frequency and mode shape of fixed-spring-mass, respectively.

The left side of the bar is fixed at a rigid wall while the right side is attached to a punctual mass $m$ and two springs fixed at a rigid wall. The first spring (of stiffness $k_1$) is linear and exerts a restoring force proportional to the stretching on the bar. The second spring (of stiffness $k_2$) is nonlinear and its restoring force is proportional to the cube of the stretching. The force which the particle exerts on the bar is proportional to acceleration. These boundary conditions read as

\begin{align*}
u(0,t) &= 0 \quad \text{and} \quad EA \frac{\partial u}{\partial x}(L,t) = -k_1 u(L,t) - k_2 \left[ u(L,t) \right]^3 - m \frac{\partial^2 u}{\partial t^2}(L,t). \quad (3)
\end{align*}

Initially, any point $x$ of the bar presents displacement and a velocity respectively equal to

$$u(x,0) = \alpha_1 \phi_3(x) + \alpha_2 x \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = 0,$$

for $0 \leq x \leq L$. In these equations $\alpha_1$ and $\alpha_2$ are constants and $\phi_3(x)$ denotes the third mode shape of fixed-spring-mass bar.
2.2 Stochastic Approach

Consider a probability space \((\Omega, \mathcal{A}, \mathbb{P})\), where \(\Omega\) is sample space, \(\mathcal{A}\) is a \(\sigma\)-field over \(\Omega\) and \(\mathbb{P}\) is a probability measure. Working in this probabilistic space, the elastic modulus is assumed to be a random variable \(E : \Omega \rightarrow \mathbb{R}\) that, associates to each event \(\omega \in \Omega\) a real number \(E(\omega)\). Consequently, the displacement of the bar becomes a random field \(U : [0, L] \times [0, T] \times \Omega \rightarrow \mathbb{R}\), which evolves according to a stochastic partial differential equation similar to Eq. (1), by changing \(u\) for \(U\), and being the partial derivatives now defined in the mean square sense [2].

The random variable \(E\) cannot assume negative values, so it is reasonable to suppose that its support is given by the interval \((0, +\infty)\). Also, it is assumed that its mean value is known. In order to the stochastic model be consistent, it is also desirable that system displacement has finite variance. Respecting these constraints, until the best of the authors knowledge, the gamma distribution is the one that most accurately describes this random parameter. This probability density function (PDF) was obtained by maximum entropy principle, as suggested in [4].

The uncertainty propagation of this random system is computed through the Monte Carlo method [3]. In this method, which is a stochastic solver, many realizations of system random parameter is generated. Each one of these realizations defines a new initial/boundary value problem given by Eqs. (1), (3), and (4), which is solved using a Galerkin procedure [1].

3 Numerical Experiments

The Monte Carlo simulation used 1024 realizations to characterize the continuous random system, which is defined by the following parameters: \(\rho = 7900\ \text{kg/m}^3\), \(A = 625\pi\ \text{mm}^2\), \(c = 10\ \text{kN} \cdot \text{s/m}\), \(L = 1\ \text{m}\), \(T = 8\ \text{ms}\), \(k_1 = 650\ \text{N/m}\), \(k_2 = 650 \times 10^{13}\ \text{N/m}^3\), \(\alpha_1 = 0.1\ \text{mm}\), \(\alpha_2 = 0.5 \times 10^{-3}\), and \(\sigma = 100\ \text{kN}\). The random variable \(E\), which the PDF is shown in Figure 2, is characterized by two parameters, the mean value \(\mu_E = 203\ \text{GPa}\) and the dispersion factor \(\delta_E = 0.1\).

![Figure 2: Probability distribution of random variable E.](image)

This continuous system is studied for three different values of the punctual mass \((m = 1.5, 7.5, \text{ and } 15\ \text{kg})\) as can be seen in Figures 3 and 4, where it is possible to see the PDF of normalized displacement and velocity at \(x = L\) and \(t = T\).

All the system analyzed present multimodal probability distribution for displacement, Figure 3, but the first two (with \(m = 1.5\) and \(7.5\ \text{kg}\)) have the modes near the extremes of random variable support, whereas the system with the greatest mass \((m = 15\ \text{kg})\) has the highest mode centered.

With respect to the normalized velocity PDF, it can be seen in Figures 4 that the mass increase tends to rise the probability of obtaining velocity values slightly right of the central position of the random variable support.
4 Concluding Remarks

This work illustrates the application of Monte Carlo method to compute uncertainty propagation of a one-dimensional elastic bar with random elastic modulus and subjected nonlinear external force. A parametric analysis revealed that, when one changes the value of the punctual mass at the right extreme of the bar, there is a change in displacement and velocity PDF modes.

References