Riemann-Stieltjes integral on time scales considering discontinuous functions and the representation of solutions with restraints in dynamical equations

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Abstract: We have yet defined in the literature of a Cauchy - Stieltjes type integral dealing with discontinuous functions in general Banach spaces on time scales. By means of this integral we can represent a linear operator as an integral. It’s possible to have the space of constraints solutions in a dynamical equation as a kernel of a linear operator, and then represent it as a constraint in the integral representation.

1 Introduction

According E. T. Bell (1883-1960), one of the main tasks in Mathematics is to be harmonizing the continuous and the discrete notions including them in an comprehensive field, eliminating in this way common obscurities.

Introduced by S. Hilger in 1988 in his Ph.D. Thesis, the calculus on time scales was founded with the purpose in to be unifying the continuous and the discrete analysis in dynamic equations. Moreover, the Calculus on time scales intend to be avoiding duplication in the analysis of separated cases because it is standing on a general T, that is a closed non-void subset of R.

By other hand when dealing with the fundamental tasks in Calculus on time scales, we must be having in mind that the results need to be working in all scales T (that could be for instance the Cantor set , the set \( \{ \frac{1}{k}, k \in \mathbb{N}^* \} \), \( \mathbb{Z}, \mathbb{R} \), and so on.

Calculus on time scales can be applied in a variety of important fields as in biology [8],[9], in Economy and in general in the field of the inequalities [7], in control theory [5], variational calculus [1] multiobjective optimization [10] and so on.

2 Differentiation on Time Scale

Let be a time scale \( T \) and consider in \( T \), the topology induced by \( \mathbb{R} \). We have two notions of finite difference: the delta: \( \Delta f(t_i) = f(t_{i+1}) - f(t_i) \) and the nabla: \( \nabla f(t_i) = f(t_i) - f(t_{i-1}) \).
providing in this way two notions of differentiability. The delta differential and the nabla one have similar operational behavior and we can only be considering the delta differentiability.

2.1 The Delta Derivative

Let be \( f : T \rightarrow \mathbb{R} \) a function and \( t \in T \). The delta derivative of \( f \) at \( t \), denoted by \( f^\Delta(t) \), is the real number (if existing) such that for every \( \epsilon > 0 \) there exists a neighborhood \( U_\epsilon^t \) in \( T \) at \( t \) with

\[
[f(\sigma(t)) - f(t)] - f^\Delta(t)[\sigma(t) - s] \leq \epsilon|\sigma(t) - s|
\]

for every \( s \in U_\epsilon^t \). We define \( \sigma(t) = \inf\{s \in T; s > t\} \).

We say that \( f \) is delta differentiable if there exists the delta derivative of \( f \) at every \( t \in T \). Observe that if \( t = \sup(T) \) or \( t = \inf(f(T)) \) we have to adapt the definition, as usual in Calculus. In the following we give the first properties for the \( f^\Delta \) derivative.

**Theorem 2.1.1** Assume \( f : T \rightarrow \mathbb{R} \) is a function and let \( t \in T \). Then we have the following:

1. If \( f \) is differentiable at \( t \), then \( f \) is continuous at \( t \).
2. If \( f \) is continuous at \( t \), and \( t \) is right-scattered, then \( f \) is differentiable at \( t \) with \( f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} \).
3. If \( t \) is right-dense, then \( f \) is differentiable at \( t \) iff the limit \( \lim_{s \to t} \frac{f(s) - f(t)}{s - t} \) exists as a finite number. In this case \( f^\Delta(t) = \lim_{s \to t} \frac{f(s) - f(t)}{s - t} \).
4. If \( f \) is differentiable at \( t \), then \( f(\sigma(t)) = f(t) + (\sigma(t) - t)f^\Delta(t) \).

**Example 2.1.1** The delta derivative of \( f(t) = t^n \), \( n \geq 1 \) for \( T = \mathbb{Z} \) or \( T = \{\frac{1}{2^k}, k \in \mathbb{N}\} \) are respectively: \( nt^{n-1} + \frac{n(n-1)}{2}t^{n-2} + \ldots + 1 \) and \( f^\Delta(t) = (2^n - 1)t^{n-1} \). Observe that for \( T = \mathbb{R} \) we get \( f^\Delta(t) = f'(t) \).

3 The Riemann-Stieltjes integral on time scales \( T \)

The Riemann integral on time scales is taken as the antiderivative operator [6]. It is more recent (2009) the work by Mozyrska-Pawlusiewicz-Torres, [12] in which the Riemann-Stieltjes integral on time scales was considered in the Darboux sense. However this type of integral presents some deficiencies when used in the frame of the discontinuous functions, as for instance when there we do not have in general that \( \int_a^b f(c) \delta = \int_a^c f(c) \delta + \int_c^b f(c) \delta \) para \( c \in (a, b) \), [11].

With the sake in to avoiding these difficulties in [3] it was defined the Cauchy-Stieltjes integral \( \int_{[a,b]T} \Delta_s \alpha(s).f(s) \), where \( f \in G([a,b]T, X) \) and \( \alpha \) is of bounded semivariation;

**Definition (semivariation):** Let be \( \mathcal{P}_{[a,b]T} \), the set of the partitions of \([a,b]T\) and \( P \in \mathcal{P}_{[a,b]T} \) and \( \alpha : [a,b]T \rightarrow L(X,W) \). The function \( \alpha \) is of bounded semivariation and we write \( \alpha \in SV([a,b]T,\mathcal{L}(X,W)) \) if

\[
SV[\alpha] = \sup_{P \in \mathcal{P}_{[a,b]T}} SV_P[\alpha]
\]

with

\[
SV_P[\alpha] = \sup \left\{ \left\| \sum_{i=1}^{\left| P \right|} [\alpha(t_i) - \alpha(t_{i-1})]x_i \right\|_W ; x_i \in X; \|x_i\| < 1 \right\}
\]

and \( SV[\alpha] \) is finite.
It is possible to represent linear operators acting on the space of the regulated functions by using the Cauchy-Stieltjes integral as shown in [3].

**Theorem 3.0.1** Let $X, W$ be Banach spaces and for every $Z \subset \mathbb{R}$, the characteristic function $\mathcal{X}_Z : \mathbb{R} \to L(X, W)$

\[
\mathcal{X}_Z = \begin{cases}
Id & \text{if } z \in Z \\
0 & \text{if } z \notin Z
\end{cases}
\]

Then

\[
\Lambda : SV_0([a, b]_T, L(X, W)) \to L(G^{-}([a, b], X), W)
\]

with $\Lambda(\alpha) = \mathcal{X}_\alpha$ is an isometry of the first space onto the second one. Further $\alpha(t)x = A(\mathcal{X}_{[a, t]}x)$. Moreover

\[
A(f) = \int_{[a, b]_T} D_s \mathcal{X}_{[t, b]}(s)f(s)
\]

When we have $W$ as a function space itself it is possible to represent the causal operators acting on the space of the regulated functions as an integral of the Cauchy-Stieltjes type by means of a kernel with two variables, as shown in [4]. We are able in this way to defining Volterra-Stieltjes equations of the second order on time scales. A huge class of dynamic equations can be represented by this Volterra-Stieltjes equations.

With the purpose in to considering in the linear case the space of the constrained solutions for these kind of equations, we represent this space as the kernel of an operator on the regulated functions. In the next section we are characterizing three of these fundamental conditions.

### 4 Constrained conditions

Let us consider the following constraints that can be imposed on the solutions in a dynamic equation on time scales: evaluation of a solution in a point ,periodicity and asymptotic stability. By using the representation for a linear operator, as done in [3] we have in all cases $A \in L(G^{-}([a, b], X), W)$:

- $A(f) = f(t_0), t_0 \in (a, b)_T$: The mapping $\alpha$ is $\alpha(t)x = A(\mathcal{X}_{[a, b]}x) = \mathcal{X}_{[t_0, b]}x$. In this way:

\[
f(t_0) = \int_{[a, b]_T} D_s \mathcal{X}_{[t_0, b]}(s)f(s).
\]

- $A(f) = f(r) - f(s) = 0$ with $r, s \in (a, b)_T$: The mapping $\alpha$ is $\alpha(t)x = A(\mathcal{X}_{[r, s]}x)$. In this way:

\[
f(r) - f(s) = \int_{[a, b]_T} D_s \mathcal{X}_{[r, s]}(s)f(s).
\]

- Consider $X = \mathbb{R}$ and $F : [a, b]_T \to \mathbb{R}$, and $A_m(f) = f(t) = m \leq f(T)$ for every $m \in [0, f(T)]$. Let be $\alpha_m(t) = \mathcal{X}_{[t, b]}$ and $A_m(f) = \int_{[t, b]_T} D_s \alpha_m(s)f(s)$, for every $m \leq F(t)$. For ending this problem consider $b \to \infty$, and $f(t) \to 0$ when $t \to \infty$. 

References


