Weak solutions for the electrophoretic motion of charged particles

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In this work we analyze the electrophoretic motion of a charged rigid particle in a viscous fluid (an electrolyte solution). The governing equation for the electrostatic potential is given by the Poisson-Boltzmann:

\[
\nabla \cdot (k(x) \nabla \psi(x)) - b(x, \psi(x)) = \rho(x), \text{ in } D, \\
\psi_0(x) = \psi_1(x), \text{ on } \partial K(t), \\
\psi(x) = \Psi(x), \text{ on } \partial D, \\
k_1 \frac{\partial \psi_1}{\partial n} - k_2 \frac{\partial \psi_2}{\partial n} = C \sigma, \text{ on } \partial K(t),
\]

where \( K(t) \in C^{1,1} \) is a domain in \( \mathbb{R}^n \) occupied by the particle in the time \( t \in [0, T] \) and \( D \in C^{1,1} \) is its enclosure; \( k : D \rightarrow L([\mathbb{R}^n, \mathbb{R}^n]), k_{ij}(x) = \delta_{ij}k_1 \) if \( x \in K(t), \) \( k_{ij}(x) = \delta_{ij}k_2 \) if \( x \in D\setminus K(t),k_1, k_2 \) are the dielectric constants of the \( K(t) \) and \( D\setminus K(t), \) \( r_D^2 \) is the Debye’s radius; \( b(x, \psi(x)) = k_2r_D^2 \sinh \psi(x) \) if \( x \in D\setminus K(t),b(x, \psi(x)) = 0 \) if \( x \in K(t), \) \( \Psi(x) \in C^1(D) \) is an external potential field; \( \rho \in L^2([0,T] \times D), \sigma \in L^2(\partial K(t)) \) are densities of fixed charges; \( C \) is a constant.

The hydrodynamics is governed by the Navier-Stokes equation for incompressible fluids:

\[
\rho_f(\partial_t v + \nabla \cdot (v \otimes v)) + \nabla p - \mu \Delta v = \rho_f f, \text{ div } v = 0 \text{ in } D\setminus K(t), \\
v = u + w \times r \text{ on } \partial K(t), \quad v = 0 \text{ on } \partial D,
\]

jointly with suitable initial conditions. Here \( \rho_f \) is the density (of the mass) of the fluid; \( \mu \) is the viscosity; \( v, p \) are the velocity and pressure of the fluid; \( u = u(t), w = w(t) \) are the translational and rotational velocities of the particle, \( r(x,t) = x - x_c(t), x_c(t) \) is the center of the mass of \( K(t); f(x,t) = (r_D^2k_2 \sinh(\psi_2(x,t)) + \rho_2(x,t))\nabla \psi_2(x,t) \) is the body force associated with the electrical action on the fluid. We are not considering the model treated in [1], [2] where the field \( -\nabla \psi_2 \) induces a relative displacement of the diffuse cloud of counterions which gives rise to a screening effect or a slip velocity of the fluid on \( \partial K(t). \)

**Lemma 1.** If \( \psi \) is the solution of (1), then \( \psi \in H^1(D) \cap C^{0,\alpha}(D), \) for some \( 0 < \alpha < 1. \)

**Lemma 2.** If \( f(x,t) = (r_D^2k_2 \sinh(\psi_2(x,t)) + \rho_2(x,t))\nabla \psi_2(x,t), \) then \( f \in L^2([0,T] \times D\setminus K(t))^n \) and \( \int_0^T \| f(t) \|^2_{0,2, D\setminus K(t)} dt \leq C, \) where \( C \) does not depend on \( (u,w). \)

With this a priori estimate for \( f \) we are able to use directly the result on the existence of appropriate weak solutions for the motion of rigid particles broached in the paper of Desjardins and Esteban [3]. Introducing the global velocity \( \tilde{u}(x,t) = v(x,t), \) if \( x \in D\setminus K(t), \) \( \tilde{u}(x,t) = u(x,t) + w(x,t) \times (x - x_c(t)) \) if \( x \in K \) and the analogous global density (of the mass) \( \rho_m \) we have, if dist(\( \partial K(t), \partial D \)) > 0, \( \forall t \in [0,T], \)

**Theorem 1.** There exist \( T^* \in (0, +\infty) \) and a weak solution \( (u, \rho_m) \) such that \( u \in L^\infty([0, T); H^1_0(D))^n \) and \( \partial_t u \in L^2([0, T) \times D)^n \) for all \( T < T^*. \)

We observe that this formulation may be adapted for the study of electroosmotic pumping in microchannels [4].

**References**


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