Summary: It is done a method for the definition of a closed continuous curve \( C \) in the plane, coming from an equation of the Liénard type, in such a way that \( C \) surrounds a periodic orbit of the equation itself.

Key words: Periodic orbits, applications of dynamical systems theory, existence of cycles.

1. INTRODUCTION

The existence of periodic orbits for Liénard equations is one of the most considered issues in the field of the nonlinear ODE in the last fifty years. This work deals with the existence of periodic orbits in a very large class (S) of nonlinear second order differential equations of Liénard generalized type including as special cases the classical Liénard ones, as well as the fractional power VdP equations and some TNO (truly nonlinear oscillators) considered by R. Mickens, [1], [2].

Through the algebraic expression concerning continuous closed curves in \( \mathbb{R}^2 \) and by comparing slopes along such curves we make considerations on the following aspects on (S):

1- the stability of singular points, and
2- the existence of periodic orbits.

The system (S) is represented by the equation

\[
\ddot{x} - f(x)g(\dot{x}) + h(x) = 0
\]  

(1)

where \( f, h \) are differentiable and \( g \) is continuous and satisfying the equation

\[
f(x)g(0) = h(x) \iff x = 0
\]  

(2)

The first order version of equation (1) in \( \mathbb{R}^2 \) is

\[
\begin{aligned}
\dot{x} &= y \\
\dot{y} &= f(x)g(y) - h(x)
\end{aligned}
\]  

(3)

According (2) the point \((0,0)\) is the unique singular point in (3).

2. PERIODIC ORBITS

2.1. The singular point and stability

A first question we ask when designing the phase portrait of (3), concerns the stability of the singular point \((0,0)\).

Classical results using linearization in (3) are well known. One of these results says, for instance, that if \( f(0)g'(0) > 2 \) then the eigenvalues in the system are strictly positive and in this way, \((0,0)\) is unstable.

It is immediate that: if there is an \( r > 0 \) such that for all \( 0 < \epsilon < r \) we have

\[
f(x)g(\sqrt{\epsilon^2-x^2}) > 0 \quad \text{in} \quad x \in (-\epsilon, \epsilon)
\]

then \((0,0)\) is an unstable singular point for (3). If otherwise

\[
f(x)g(\sqrt{\epsilon^2-x^2}) < 0
\]

(0,0) is a stable one. In fact: under the hypothesis, we get \( f(x)g(y) > 0 \), for all \( C_{\epsilon} = C_{\epsilon}(x,y) \), the semi-circle centered at \((0,0)\) with radius \( \epsilon \) and \( y > 0 \). This fact implies that

\[
f(x)g(y) - \frac{x}{y} > -\frac{x}{y}
\]

(4)

on \( C_{\epsilon} \). By observing that the first term in the inequality in (4) is the slope \( \frac{dy}{dx} \) of the orbit for (3) through the point \((x,y)\) and that \( \dot{x} = 0 \), then all orbits for (3) crosses the curve \( C_{\epsilon} \cup \{(-\epsilon,0),(\epsilon,0)\} \) from inside [respect to the point \((0,0)\)] to outside, showing in this way that the singular orbit \((0,0)\) is repulsive.
From now on we consider the unique singular point (0,0) in the system being unstable.

2.2. Existence of periodic orbits

If for all \((x, y) \in \mathbb{R}^2\),

\[
\sigma^2 = x^2 + y^2.
\]

we have along the orbits of (3):

\[
\frac{\partial \sigma}{\partial x} = f(x)g(y) + x - h(x)
\]

(5)

Let us start defining the closed continuous curve \(C = C(x,y)\) \((y \geq 0)\), crossing the axis \(y = 0\) at the points \(x = r\) and \(x = -r\) \((r > 0)\), enclosing (if any) at least one periodic orbit of (3).

Consider

\[
\rho^2 = x^2 + y^2 \quad \text{on} \quad (x, y) \in C.
\]

If \(M > 0\) then there exists a number \(S_M \geq 0\) such that \(\max_{\text{sys} \mathcal{M}} g(y) \leq S_M\).

Let us make \(C = C(x,y)\) \((y \geq 0)\), obeying the equation

\[
\begin{cases}
\dot{x} = y \\
\dot{y} = S_Mf(x) - h(x)
\end{cases}
\]

(6)

We have immediately, that at the points \((x, y) \in C\),

\[
\frac{\rho^2(x)}{2} = S_MF(x) + \frac{x^2}{2} - H(x) + c
\]

(7)

with \(c\) constant and \(F(x), H(x)\) being respectively the anti-derivative of \(f(x), h(x)\) with independent constant terms equal to zero.

If an orbit \(y\) of (3) crosses the curve \(C = C(x,y)\) \((y \geq 0)\), then looking at (5) and (7) we have \(y\) crossing \(C\) from the “outside” part of the region enclosed by \(C\) (respect to \((0,0)\)) directed into the “inside” one.

Due to the fact that \((0,0)\) is unstable then by the Poincaré-Bendixon theorem, the existence of such \(C\) shows that there exists at least one periodic orbit of (3) enclosed by \(C\).

With the aim of be showing the existence of such curve \(C\) we have to determine the constants \(r, S_M\) and \(M\) involved in the above process.

2.3 The constants \(r, S_M\) and \(M\)

Because the curve \(C\) begins at \((-r, 0)\) and ends at \((r, 0)\) we have according (7), both:

\[
r^2 = 2S_MF(-r) + r^2 - 2H(-r) + 2c
\]

and

\[
r^2 = 2S_MF(r) + r^2 - 2H(r) - 2S_MF(-r) + 2H(-r)
\]

releasing in this way \(r > 0\) and \(S_M\) related by the equality:

\[
H(r) - H(-r) = S_M(F(r) - F(-r)).
\]

(8)

Let us give now, an estimate for a possible value for \(S_M\) (and ultimately for \(M\)).

If we use (8) in (7) then \(\rho(x)\) is:

\[
\frac{\rho^2(x)}{2} = S_MF(x) + \frac{x^2}{2} - H(x) - S_MF(-r) + H(-r)
\]

(9)

where \(r > 0\) (still depending on \(S_M\)) is a point satisfying (8).

From (9) we get:

\[
\frac{d}{dx} \rho^2(x) = 2S_Mf(x) + 2x - 2h(x)
\]

(10)

and

\[
\frac{d^2}{dx^2} \rho^2(x) = 2S_Mf'(x) + 2 - 2h'(x)
\]

(11)

From (10 and (11) we can give an estimate for \(S_M\) : it will be a number for which it is possible to choose a \(x_0 \in [-r, r]\) with \(\rho(x_0) \geq 0\) in (9) with

\[
S_Mf(x_0) + 2x_0 - h(x_0) = 0
\]

and

\[
S_Mf'(x_0) + 2 + h'(x_0) > 0.
\]

Finally, \(M > 0\) is a number for which

\[
\max_{\text{sys} \mathcal{M}} g(y) \leq S_M.
\]

Gathering all the previous results it is possible to conclude that if \((0,0)\) is the unique singular point in (3) and further, it is repulsive and \(f, g, h\) are functions that allows the existence of the constants \(r, S_M\) and \(M\), then equation (1) possess at least one non-trivial periodic orbit \(\Gamma\) in \(R^2\) and this orbit is enclosed by \(C = C(x,y)\) \((y \geq 0)\). The expression (9) gives the equation for \(C\).

Elsewhere by using different equations but under the same point of view that in this paper, B.C. Damasceno [3] defined a sequence of curves in \(R^2\) approaching indefinitely \(\Gamma\).

CONCLUSION

It was proposed in this paper a very simple method (in the sense that we only used the fundamental and almost naïve analysis on crossing continuous curves in the plane) in which sufficient conditions for the existence of periodic orbits in a class of Liénard equations it was done.

The applicability of the results are related ultimately to choose well defined constants \(r, S_M\) and \(M\) in the process. For instance, if:

\[
f(x) = 1 - x^2, \quad g(y) = y^\frac{1}{2}, \quad h(x) = x^\frac{2}{3}
\]
we have $r = 2$ and $S_M = \frac{7}{5}$ being working in this case. For every even functions $f, h$ and $g(x), x > 0$ we can determine $r, S_M$ and $M$ for the existence of $C$ in an effective way, too. Notice that in [4] most profound results about the issue are done.

REFERENCES


