A STUDY OF PREY-PREDATOR PROBLEM WITH COMPETITION

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Abstract: Usually the prey-predator and competition problems are considered separately. Nevertheless when many species cohabit the same environment it is natural that these two problems occur simultaneously. This work analyzes a prey-predator-competition dynamical system. The focus of the study is the change of stability of particular three equilibrium points, and the dynamic of the populations is analyzed with numerical simulations.

Key words: Prey-predator, competition problem, application of dynamical systems.

1. INTRODUCTION

The conventional model of competition between two species is known as Gause-Lotka-Volterra dynamical system, and it is given by

$$\dot{y}_i = y_i (r_i - a_{ij} y_j - a_{il} y_l)$$

This problem is easily generalized for \( n \) competing populations as

$$\dot{y}_i = y_i (r_i - \sum_{j=1}^{n} a_{ij} y_j), \quad i = 1,2,\cdots, n,$$

and May and Leonard [1] presented a study of the case \( n = 3 \) making some simplifications. They give emphasis to limit cycle solutions.

Afraimovich, Hsu and Lin [2] analyzed the May-Leonard system introducing small periodic perturbations. Depending on some parameters the system presents chaotic regimes. Another study of the system was performed by Leach and Miritzis [3] with the approaches of singularity and symmetry analysis.

2. COMPETITION PROBLEM WITH PREDATOR

Let be \( y_1 \) and \( y_2 \) the prey populations and \( y_3 \) the predator population. Using the classical formulation the differential equations of the problem are given by

$$\dot{y}_1 = y_1 (r_1 - c \alpha y_1 - a_1 y_2 - a_2 y_3) = f_1(y_1, y_2, y_3)$$

$$\dot{y}_2 = y_2 (r_2 - b \beta y_2 - b_1 y_1 - b_2 y_3) = f_2(y_1, y_2, y_3)$$

$$\dot{y}_3 = y_3 (-r_3 + c_1 y_1 + c_2 y_2) = f_3(y_1, y_2, y_3)$$

This formulation requires 11 positive parameters and the analysis is done without simplification hypothesis.

2.1 Equilibrium points

This prey-predator-competition (ppc) problem has 7 equilibrium points. The first one is the trivial equilibrium point \( P_1 = (0,0,0) \), and the others are

$$P_2 = (0, \frac{r_1}{c_2}, \frac{s_1}{b_2}); \quad P_3 = (\frac{r_1}{c_1}, 0, \frac{s_1}{a_2});$$

$$P_4 = (\frac{n_1}{d}, \frac{n_2}{d}, 0); \quad P_3 = (0, \frac{r_1}{\beta}, 0);$$

$$P_6 = (\frac{r_1}{\alpha}, 0,0)$$

where

$$s_3 = r_2 - \beta \frac{r_1}{c_2}; \quad s_3' = r_1 - \alpha \frac{r_1}{c_1};$$

$$n_1 = \beta r_1 - a_1 r_2; \quad n_2 = \alpha r_2 - b_2 r_1;$$

$$d = \alpha \beta - a_1 b_1.$$

The magnitudes of the coefficients must be such that \( s_3 > 0, \quad s_3' > 0, \quad n_1 > 0 \) and \( d > 0, \quad n_1 < 0 \) and \( d < 0, \quad n_2 > 0 \) and \( d > 0 \), and \( n_2 < 0 \) and \( d < 0 \). The last equilibrium point is \( P_7 = (y_1^*, y_2^*, y_3^*) \) with

$$y_1^* = \frac{1}{\alpha} (r_1 - a_1 y_2^* - a_2 y_3^*)$$

$$y_2^* = \frac{1}{y_1^*} \left[ r_2 - \frac{b_1}{\alpha} r_1 + \left( \frac{b_1}{\alpha} a_2 - b_2 \right) y_3^* \right]$$

$$y_3^* = \frac{1}{y_2^*} \left[ -r_3 + c_1 y_1^* + c_2 y_2^* \right]$$
where \( \gamma = \beta - \frac{b_1}{\alpha} a_1 \).

### 2.2 Jacobian matrix

Taking vector notation \( y = (y_1, y_2, y_3)^T \) and \( f = (f_1, f_2, f_3)^T \), the linear dynamical system associated to equilibrium point \( P \) is given by

\[
\dot{y} = \frac{df}{dy} \bigg|_P (y - P)
\]

where \( J \) is the Jacobian matrix which elements are

\[
J_{11} = r_1 - 2\alpha y_1 - a_1 y_2 - a_2 y_3; \quad J_{12} = -a_1 y_1; \quad J_{13} = -a_2 y_1;
\]

\[
J_{21} = -b_1 y_2; \quad J_{22} = r_2 - 2\beta y_2 - b_1 y_1 - b_2 y_3; \quad J_{23} = -b_2 y_2;
\]

\[
J_{31} = c_1 y_3; \quad J_{32} = c_2 y_3; \quad J_{33} = -r_3 + c_1 y_1 + c_2 y_2.
\]

### 3. ROUTH-HURWITZ CRITERION

In this section the stability of equilibrium points \( P_2 \) and \( P_3 \) is analyzed by Routh-Hurwitz criterion, because these points change the stability at critical values of control parameters.

#### 3.1 Analysis of point \( P_2 \)

The coordinates of \( P_2 \) are \((0, \frac{r_3}{c_2}, \frac{s_2}{b_2})\) (see Equations (2)), then the Jacobian matrix is given by

\[
J_2 = \begin{pmatrix}
  s_1 + s_3 \frac{r_5}{c_2} & 0 & 0 \\
  -b_1 \frac{r_5}{c_2} & -b_2 \frac{r_5}{c_2} & -b_2 \frac{r_5}{c_2} \\
  c_1 \frac{s_3}{b_2} & c_2 \frac{s_3}{b_2} & 0
\end{pmatrix}
\]

with \( s_1 = r_1 - \frac{a_2}{b_2} r_2 \), \( s_2 = \frac{a_2}{b_2} \beta - a_1 \) and \( s_3 \) given in Equations (3). The characteristic polynomial obtained from \( \text{det}(J_2 - \lambda I) = 0 \) is expressed as

\[
\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0
\]

which coefficients are

\[
A_1 = \beta \frac{r_5}{c_2} - S_1; \quad A_2 = r_5 s_3 \frac{r_5}{c_2};
\]

\[
A_3 = r_3 s_3 S_1; \quad S_1 = s_1 + s_2 \frac{r_5}{c_2}
\]

Then it follows that the Hurwitz matrix is

\[
H = \begin{pmatrix}
  A_1 & A_2 & 0 \\
  1 & A_2 & 0 \\
  0 & 1 & A_3
\end{pmatrix}
\]

and all eigenvalues of Jacobian matrix \( J_2 \) have negative real part under the conditions

\[
A_i > 0, \quad i = 1, 2, 3; \quad \text{det}(H) > 0; \quad \text{det}(H) > 0
\]

Because \( P_2 \) is independent of \( r_1 \), this coefficient is taken as control parameter. The above conditions expressed in terms of this parameter become

\[
A_i > 0 \iff r_1 < \frac{a_2}{b_2} r_2 + \frac{r_1}{c_2} (\beta - s_2)
\]

\[
A_2 > 0 \iff r_1 < \frac{a_2}{b_2} r_2 - \frac{r_1}{c_2} s_2 + \frac{c_2}{\beta} s_3
\]

\[
A_3 > 0 \iff r_1 < \frac{a_2}{b_2} r_2 - \frac{r_1}{c_2} s_2
\]

\[
det(H) > 0 \iff r_1^2 + Q r_1 + R > 0
\]

\[
det(H) > 0 \iff -r_1^2 + U r_1^2 + V r_1 + W > 0
\]

where

\[
Q = M - 2 \frac{a_2}{b_2} r_5, \quad R = \left( \frac{a_2}{b_2} r_2 \right)^2 - \frac{a_2}{b_2} r_5 M - N
\]

\[
M = r_1 (2 s_2 - \beta) s_1, \quad N = \left( \frac{r_1}{c_2} s_2 \right)^2 - \beta \left( \frac{r_1}{c_2} \right)^2 s_2 + r_5 s_3
\]

\[
U = \frac{a_2}{b_2} r_2 - \frac{r_1}{c_2} s_2 - Q, \quad V = \left( \frac{a_2}{b_2} r_2 - \frac{r_1}{c_2} s_2 \right) Q - R
\]

\[
W = \left( \frac{a_2}{b_2} r_2 - \frac{r_1}{c_2} s_2 \right) R.
\]

#### 3.2 Analysis of point \( P_3 \)
According Equations (2) the equilibrium point \( P_3 \) has coordinates \((\frac{r_a}{c_1}, 0, \frac{r_c}{a_2})\), then the Jacobian matrix \( J_3 \) becomes
\[
J_3 = \begin{pmatrix}
-\alpha \frac{r_a}{c_1} & -a_1 \frac{r_a}{c_1} & -a_2 \frac{r_a}{c_1} \\
0 & \frac{r_a}{c_1} \frac{\alpha}{a_2} - \frac{b_1}{a_2} & \frac{r_c}{a_2} \\
\frac{r_a}{a_2} & \frac{r_c}{a_2} & \frac{r_a}{a_2} \frac{\alpha}{a_2} - \frac{b_1}{a_2}
\end{pmatrix}
\] (7)

with \( s_1' = r_2 - \frac{b_2}{a_2} r_1, \ s_2' = \frac{b_2}{a_2} \alpha - b_1 \) and \( s_3' \) is given in Equations (3). Now the coefficient \( r_2 \) is taken as control parameter and a similar study determines the critical value for this parameter.

4. NUMERICAL SIMULATIONS

Once established the conditions for critical values of control parameters, the values given in the Table 1 for coefficients are adopted to analyze the dynamical system.

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Value</th>
<th>Coefficient</th>
<th>Value</th>
</tr>
</thead>
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<tr>
<td>( a_1 )</td>
<td>0.07</td>
<td>( c_1 )</td>
<td>0.03</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>0.1</td>
<td>( c_2 )</td>
<td>0.05</td>
</tr>
<tr>
<td>( b_1 )</td>
<td>0.05</td>
<td>( \alpha )</td>
<td>0.01</td>
</tr>
<tr>
<td>( b_2 )</td>
<td>0.06</td>
<td>( \beta )</td>
<td>0.015</td>
</tr>
<tr>
<td>( r_3 )</td>
<td>0.5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Taking the value \( r_2 = 1.5 \), the conditions (6) give \( r_1 < 2.95 \) to keep the equilibrium point \( P_2 \) stable. Assuming \( r_1 = 2.0 \) the equilibrium point \( P_3 \) is stable for \( r_2 < 1.93333 \). Fixing \( r_1 = 2.0, \ r_2 = 1.5 \) and considering the values of Table 1, the coordinates of equilibrium point \( P_3 \) are \( (0, 10, 22.5) \). This point is stable because the eigenvalues of Jacobian matrix \( J_2 \) given in Equation (5) are
\[
\lambda_1 = -0.95 \\
\lambda_2 = -0.075 + 0.8182i \\
\lambda_3 = -0.075 - 0.8182i
\] (8)

In a same way, the equilibrium point \( P_3 \) has coordinates \((16.6667, 0, 18.3333)\) and the eigenvalues of \( J_3 \) (Equation (7)) are
\[
\lambda_1 = -0.0833 + 0.9538i \\
\lambda_2 = -0.4333 \\
\lambda_3 = -0.0833 - 0.9538i
\] (9)

Figure 1 shows the evolution of populations \( y_2 \) and \( y_3 \) from initial conditions \((y_{10}, y_{20}, y_{30}) = (5.0, 5.0, 5.0)\), and after \( t = 50 \) units it is clear that the solution \((y_1, y_2, y_3)\) tends to \( P_2 \). Now, taking the initial conditions \((y_{10}, y_{20}, y_{30}) = (1.0, 1.0, 1.0)\), the solution tends to \( P_3 \) as indicates Figure 2.
Considering the above mentioned values of parameters, the coordinates of equilibrium point $P_1$ are $(9.4684, 4.3189, 16.0299)$, and the eigenvalues of $J_1$ are $\lambda_1 = 0.2979$ and $\lambda_{2,3} = -0.2287 \pm 0.7810i$. Therefore this point is unstable. For $r_1 > 2.95$ the eigenvalue $\lambda_1$ of $J_2$ becomes positive and similarly for $r_2 > 1.93333$ the eigenvalue $\lambda_2$ of $J_3$ is positive.

Nevertheless it is interesting to mention that for $r_1 = 2.951$ the point $P_1$ assumes coordinates $(-0.00997, 10.00598, 22.50681)$ and eigenvalues $\lambda_1 = -0.001$ and $\lambda_{2,3} = -0.0745 \pm 0.8183i$, very close to the values given in (8). This result means that $P_1$ occupies the position of $P_2$ and is stable.

Otherwise for $r_2 = 1.934$ the equilibrium point $P_2$ becomes $P_2 = (16.6777, -0.0066, 18.3369)$, and the eigenvalues are $\lambda_2 = -0.0067$ and $\lambda_{3,4} = -0.0845 \pm 0.9524i$, now very close to the values given in (9). In this case $P_2$ coincides with $P_1$.

Figure 3 shows the dynamic of populations $y_1$ and $y_3$ from initial conditions $(y_{10}, y_{20}, y_{30}) = (10.0, 1.0, 1.0)$ and $r_2 = 1.94$. After $t = 100$ units the solution $(y_1, y_2, y_3)$ is $(16.5300, 0.0762, 18.2913)$.

The analysis of ppc dynamical system showed that while $r_1 < 2.95$, $P_2$ is stable and $P_1$ is unstable. At the critical value $r_1 = 2.95$ these two points coalesce. For $r_1 > 2.95$ the point is stable but the coordinate $y_1$ becomes negative. Similarly for $r_2 < 1.93333$ $P_3$ is stable and $P_1$ is unstable. At the critical value these two points coincide and for $r_2 > 1.93333$ the point is stable but the coordinate $y_2$ becomes negative. These results indicate interesting bifurcation phenomena that must be analyzed using adequate methodologies.

**REFERENCES**


**5. DISCUSSION**

The ppc problem modeled conventionally presents 7 equilibrium points, namely $P_1$ to $P_7$. $P_2$ is independent of Malthusian growth rate $r_1$ and $P_3$ is independent of $r_2$. So these parameters were used as control parameter to analyze the stability of the equilibrium points $P_2$, $P_3$ and $P_7$. The critical values were determined by Routh-Hurwitz criterion.

**6. CONCLUSION**