STABILIZATION BY OUTPUT FEEDBACK IN REGIONAL POLE PLACEMENT

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1. INTRODUCTION

The problem of static output feedback by regional pole placement in Linear matrices inequalities (LMI) regions is considered in this paper. A new approach for the design of output feedback controller is proposed and the respective output feedback gains are obtained through the solution of the coupled Sylvester equation which is related to the solution of Lyapunov equations that allow to obtain pole placement in LMI regions of the complex plane. It presented necessary and sufficient conditions for the existence of the static output feedback matrix. This paper also proposes a new numerically efficient solution algorithm for the coupled design equations to determine the output feedback matrix. Finally, many theoretical conditions have been offered for the existence of output feedback matrix.

This paper is organized as follows. The second section presents the problem based on the basic concepts and LMI regions basic concepts. In section 3, we present some results in the form of Sylvester equations, the Theorem for the existence of the output feedback matrix. Numerical examples illustrate the application of the algorithms which outline the basic steps used to solve the problem. They are presented in the fourth section. Finally, the concluding remarks are presented.

2. PRELIMINARIES

The solutions to the problem stabilization is associated to the existence of a subspace $V = \text{Ker}(T)$ O.S. $(C, A, B)$ invariant, due to the possibility of solving the coupled Sylvester equations, rewritten below:

$$AV + VH = -BW \text{ with } \sigma(H) \subseteq C^-$$

$$TA + HT = -UC \text{ with } \sigma(H) \subseteq C^-$$

$$TV = 0$$

$$\text{Ker}(CV) \subseteq \text{Ker}(W)$$

$$\text{Ker}(B'T') \subseteq \text{Ker}(U')$$

In [5], the coupled equations Lyapunov can also be used to describe geometrical properties required for solving the stabilization problem.

Theorem 2.1: [5] There exists an output feedback matrix $G \in \mathbb{R}^{m \times p}$ such that $\sigma(A + BGC) \subseteq C^-$, if and only if the conditions verified for some full-rank matrices $V \in \mathbb{R}^{n \times v}$ and $T \in \mathbb{R}^{v \times n}$, with integer $0 < v \leq n$, such that $TV = 0$:

$(i) \forall Q_V = Q'_V > 0$, $Q' \in \mathbb{R}^{v \times v}$, there exist the matrices $P = P' \geq 0$, $P \in \mathbb{R}^{n \times n}$ and $Y \in \mathbb{R}^{m \times n}$ such that:

$$AP + PA' + BY + Y'B' = -QVvV'$$

$$V'PV > 0; \quad TP'T' = 0$$

$$Y = W_{II}V' \text{ for some } W_{II} \in \mathbb{R}^{m \times v}$$
(ii) \( \forall Q_T = Q_T' > 0, Q_T \in \mathbb{R}^{n \times n} \), there exist the matrices \( S = S' \geq 0, S \in \mathbb{R}^{n \times n} \) and \( Z \in \mathbb{R}^{n \times v} \) such that:

\[
A'S + SA + C'Z' + ZC = -T'Q_T T \quad (9)
\]

\[
TST' > 0 ; V'SV = 0 \quad (10)
\]

\[
Z = T'U_T \text{ for some } U_T \in \mathbb{R}^{n \times v} \quad (11)
\]

(iii)

\[
\text{Ker} CP \subseteq \text{Ker} Y
\]

\[
\text{Ker} B'S \subseteq \text{Ker} Z' \quad (12)
\]

\[
\text{Ker} B'S \subseteq \text{Ker} Z' \quad (13)
\]

3. REGIONAL POLE PLACEMENT

A solution for the stabilization problem to obtain output feedback matrix was based on the arbitrary pole placement. There are solutions for the stabilization problem considering the displacement of \( \alpha \) using \((A + \alpha I)\) instead of \( A \). The possibility to stabilize and situate the poles in a determined \( LMI \) region \( D \) as in [9].

Let \( D \subset C \) some region of the complex plane. The stabilization problem is to find an output feedback matrix \( u = Gy \) such that \( \sigma(A + BGC) \in D \), or equivalently, the closed loop system is \( D \)-stable.

A pair \((A, B)\) is \( D \)-stabilizable if and only if

\[
\text{rank} \left[ \begin{array}{cc} \lambda I - A & B \\ \end{array} \right] = n, \quad \forall \lambda \notin D \quad (14)
\]

and a pair \((C, A)\) is \( D \)-detectable if and only if

\[
\text{rank} \left[ \begin{array}{cc} \lambda I - A & C \\ \end{array} \right] = n, \quad \forall \lambda \notin D \quad (15)
\]

3.1. LMI regions basic concepts

The \( LMI \) regions describe convex regions in complex plane that are symmetric in relation to the real axis.

**Definition 3.1**: A subset \( D \)

\[
D = \{ \lambda \in \mathbb{C} : f_D(\lambda) < 0 \} \quad (16)
\]

complex plane is called a \( LMI \) region if there is a symmetric matrix \( \Delta = (\delta)_{kl} \in \mathbb{R}^{n \times n} \) and a matrix \( \Theta = (\theta)_{kl} \in \mathbb{R}^{n \times n} \) such that

\[
f_D(\lambda) = \Delta + \lambda \Theta + \bar{\lambda} \Theta' \quad (17)
\]

with \( \lambda, \bar{\lambda} \in \mathbb{C} \), where \( C \) denotes the complex half-plane.

The characteristic function \( f_D(\lambda) \) takes values in the space of Hermitian matrices of dimensions \( n \times n \). Linear matrices inequalities, \( (LMI) \)s can be defined from \( f_D(\lambda) \) to characterize the properties of the eigenvalues of a real matrix belonging to \( D \) [9].

The \( LMI \) regions with their characteristic functions are given in [9].

- A disk-centric \((-q, 0)\) with radius \( r \), whose characteristic function is:

\[
f_D(\lambda) = \left( \begin{array}{cc} -r & q \\ q & -r \\ \end{array} \right) + \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \lambda + \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \bar{\lambda} \quad (18)
\]

Consider the linear system represented by

\[
\dot{z}(t) = Hz(t), \quad \text{with } H \in \mathbb{R}^{n \times n} \quad (19)
\]

The system (19) is \( D \)-stable, where \( D \) is a \( LMI \) region, if and only if there exist matrices \( \Delta \) and \( \Theta \in \mathbb{R}^{n \times n} \)

\[
f_D(\Theta) = \Delta + \Theta \otimes H + \Theta' \otimes H' < 0 \quad (20)
\]

where \( \otimes \) represents the Kronecker product in [12].

It is interesting to note that:

- the equation (20) can be rewritten as

\[
f_D(\theta) = (\delta)_{kl} + (\theta)_{kl}H + (\theta)_{kl}H', \quad \forall 1 \leq k, l \leq n \quad (21)
\]

- for two \( LMI \) regions, \( D_1 \) and \( D_2 \), and whose characteristics functions are associated to \( f_{D_1} \) and \( f_{D_2} \), respectively, the intersection \( D = D_1 \cap D_2 \) is an \( LMI \) region with characteristic function \( f_{D_1 \cap D_2} = \text{diag}(f_{D_1}, f_{D_2}). \)

3.2. \( D \)-Stabilization by output feedback

Remember first that the existence of an output feedback matrix that stabilizes the closed loop system can be associated with to solution of coupled Sylvester equations (1), (2) and (3). By Theorem in [5], these equations can be interpreted in terms of geometric properties of the subspace \( V = \text{Ker} (T) \). Once found the output feedback matrix which stabilizes the system, the spectrum of the closed loop is given by:

\[
\sigma(A + BGC) = \sigma(H_V) \cup \sigma(H_T) \quad (22)
\]

Then, in sequence it is considered that the desired regional pole placement is defined from \( LMI \) regions associated to the matrices \( H_V \) and \( H_T \), as follows:

\[
\sigma(H_V) \in \mathbb{D}_V \implies D = D_V \cup D_T \quad (23)
\]

Then, assuming that \( H_V \in \mathbb{R}^{p \times p} \) and \( H_T \in \mathbb{R}^{n-p \times n} \), \( D \)-stabilizability conditions in \( LMI \) regions are equivalently replaced by the following two conditions:

\[
\sigma(H_V) \in \mathbb{D}_V \iff \exists \Pi = I' > 0 \text{ such that } \\
\Delta_V \otimes \Pi + \Theta_V' \otimes (\Pi H_V') + \Theta_V \otimes (H_V \Pi) = -Q_V < 0 \quad (24)
\]

where \( \Pi \in \mathbb{R}^{p \times p} \)

\[
\sigma(H_T) \in \mathbb{D}_T \iff \exists \Gamma = I' > 0 \text{ such that } \\
\Delta_T \otimes \Gamma + \Theta_T' \otimes (H_T \Gamma) + \Theta_T \otimes (H_T \Gamma) = -Q_T < 0 \quad (25)
\]

where \( \Gamma \in \mathbb{R}^{n-p \times n} \).

The following result can be obtained similarly to the result in the previous section, considering as starting point the coupled Sylvester equations under the regional pole placement restricted to the spectrum of matrices \( H_V \) and \( H_T \), respectively. For algorithmic purposes, it is considered that \( v = p \). The necessary and sufficient conditions are presented for the existence output feedback matrix given by the following theorem.

**Theorem 3.1**: There exists a static output feedback matrix \( G \in \mathbb{R}^{m \times p} \) such that \( \sigma(A + BGC) = \{\lambda_1, \lambda_2, ..., \lambda_n\} \in D \), for \( D = D_V \cup D_T \), where \( \{\lambda_1, ..., \lambda_n\} \in D_V \) and \( \{\lambda_{n-p+1}, ..., \lambda_n\} \in D_T \), if and only if there exist matrices \( P \in \mathbb{R}^{n \times n} \), \( Y \in \mathbb{R}^{n \times n} \), \( S = S' \geq 0, S \in \mathbb{R}^{n \times n} \) and \( Z \in \mathbb{R}^{n \times p} \) such that the following conditions are verified for any pair of matrices \( V \in \mathbb{R}^{n \times p} \) and \( T \in \mathbb{R}^{n-p \times n} \) such that \( TV = 0 \):

\[
\Delta_V \otimes P + \Theta_V \otimes (AP + BV) + \Theta_T' \otimes (PA' + Y'B') = -(V \otimes I_p)Q_V (V \otimes I_p)' \quad (26)
\]
\[ V'PV = \tilde{P} > 0 ; \quad TPT' = 0 \]  
(27)

\[ Y = \Pi V' \quad \text{for some} \quad \Pi \in \mathbb{R}^{m \times p} \]  
(28)

\[ \Delta_T \otimes S + \Theta_T \otimes (SA + ZC) + \Theta_T' \otimes (AS' + C'Z') = - (T' \otimes H_{II})Q_{T1}(I_{m-p} \otimes T) \quad \text{for} \quad Q_{T1} > 0 \]  
(29)

\[ TST' = \tilde{S} > 0 ; \quad V'SV = 0 \]  
(30)

\[ Z = T'U_T \quad \text{for some} \quad U_T \in \mathbb{R}^{m \times p} \]  
(31)

**Proof:**

**Necessity:** Consider that the coupled Sylvester equations (1),(2) and (3) are verified with the conditions (4) and (5).

It first shows the necessity part (i). Since the quadratic condition given by (24) must be true, and using the definition of Kronecker product \(^2\):

\[ (\delta_V)_{kl}(VIIV') + (\theta_V)_{kl}(VIIH_{II}'V') + (\theta_V)_{kl}(VH_{IV}IV') = -V(Q_{kl})V' \quad \forall k, l \leq n \]  
(34)

where: \((Q_{V})_{kl} = (Q_{V})_{kl} \in \mathbb{R}^{p \times p}.

From (1), it is also obtained \(AV + BW = VH_V\), that can be used in (35) to obtain

\[ (\delta_V)_{kl}(VIII'V') + (\theta_V)_{kl}(VIIH_{II}'V') + (\theta_V)_{kl}(VH_{IV}IV') = \]  
(35)

\[ (\theta_V)_{kl}(AVIIV' + BWIII') = -V(Q_{kl})V' \quad \forall k, l \leq n \]

Thus, by placing the matrices \(P = P' = VIIV'\) and \(Y = WIIV'\), and considering that the \(rank(V) = p\) and that \(\Pi > 0 \quad \implies \quad \{ V'V'V'V > 0 \}\)

\[ TTV'V'T' = 0 \]

(36) can be equivalently replaced by (26), (27), and (28).

Using similar arguments, it shows the necessity part (ii). Thus, from (2) and (25), it is obtained:

\[ (\delta_T)_{kl}(T'TTT) + (\theta_T)_{kl}(T'H_{II}TT) + (\theta_T)_{kl}(TT'H_{II}T) = \]  
(36)

\[ -T'(Q_{kl})_T \quad \forall k, l \leq n \]

where: \((Q_{T})_{kl} = (Q_{T})_{kl} \in \mathbb{R}^{n \times n \times n \times p}.

\[ TTT' = 0 \]

For the matrices \(S = TTT'\) and \(Z = TTV'\), (37) may be replaced by (29) and (31). Moreover, since \(rank(T) = q - p\) and \(T > 0\), it is obtained

\[ TTV' = 0 \]

The necessity of part (iii), (32) and (33) follow the conditions \(KerCV \subseteq KerW\) and \(KerB'S \subseteq KerU\), respectively, taking into account the definitions above. \(P\), \(Y\), \(S\) and \(Z\).

**Sufficiency:** It is considered that the part (i), (ii) and (iii) are verified.

### 4. OUTPUT FEEDBACK STABILIZATION ALGORITHM

#### 4.1. Primal Algorithm

Based on the primal algorithm proposed in the previous section, the following basic procedure is proposed to compute a \(D\)-stabilizing static output feedback matrix in the closed loop system when the condition \(m+p > n\) is verified:

**Step 1:**

1.1) Find a decomposition

\[ C \begin{bmatrix} M_1 & M_2 \end{bmatrix} = \begin{bmatrix} C_1 & 0 \end{bmatrix} \]  
(37)

where: \(C_1 \in \mathbb{R}^{p \times p}, rank(C_1) = p\).

To compute matrices \(A_{12}, A_{22}:

\[ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} M_1' & M_2' \\ 0 & 0 \end{bmatrix} A \begin{bmatrix} M_1 & M_2 \end{bmatrix} \]  
(38)

1.2) Solve the \(LMI\) to find \(S_{21}\) and \(S_{22} = S'_{22} > 0:

\[ \Delta_T \otimes (S_{22}) + \Theta_T \otimes (S_{22}A_{22} + S_{21}A_{12}) + \Theta_T' \otimes (A_{22}S_{22} + A'_{21}S_{21}) < 0 \]

(39)

1.3) Compute \(T = [S_{21}, S_{22}] \begin{bmatrix} M_1' & M_2' \end{bmatrix}\). If \((A, B, T)\) has no invariant zeros, go to Step 2; or else, repeat the Step 1 using a new decomposition for \(C\).

**Step 2:**

2.1) Compute \(V\) as an orthogonal basis of \(Ker(T)\):

\[ TV = 0 \quad \text{with} \quad V'V = I_p \]  
(40)

2.2) Solve the following equation, under the constraints (27) and (28), to find \(P\) and \(Y\):

\[ \Delta_V \otimes P + \Theta_V \otimes (AP + BY) + \Theta_V' \otimes (PA' + Y'B) = -V \otimes I_p Q_{V1}(V \otimes I_p)' \leq 0 \]  
(41)

**Step 3:** The \(D\)-stabilizing output feedback matrix in the corresponding closed loop system verifies

\[ GCP = Y \iff GCV \tilde{P} = \Pi, \quad \text{since} \quad V'V = I_p \]

**Remark 4.1** Note also that equation (40) is an \(LMJ\) in the variables \(S_{21}\) and \(S_{22}\). Then, convex programming techniques can be used to find feasible solutions in [2]. As a feasibility test for step 2, you should verify if the system \((A, B, T)\) has no invariant zeros which belong to \(D_V\). Then go to the next step. The conditions are usually solved in the sense that the matrix system \(F \in [A - \lambda M_n B \quad T] \quad \text{of dimension} \quad (n + m) \times (2n - p) \quad \text{must have full row rank} \quad \forall \lambda\).

Thus, the Step 2 is only feasible with stable solutions.

### 4.2. Examples

Consider the system \((A, B, C)\) with data in [11]:

\[ A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

The corresponding system \((C, A, B)\) is stabilizable and detectable, and \(m+p = 5 > n\). The open loop poles are: 0.0000, 0.0000, 1.6180, -0.6180.

**Primal Algorithm:**

The \(LMI\) region \(D_T\) is a vertical band defined by: \(D_T = \{ x + yj \in C : -5 < x < -1 \}\). The \(LMI\) region \(D_V\) is defined from the intersection of sectors and a conical disk defined by: \(\{ x + yj \in C : x < -5, |x+yj| < 10 \text{ and tan} \ 45^\circ x < |y| \}\). Thus, convex programming techniques are applied to find feasible solutions for the equations of regional pole placement without additional requirements.

The matrix \(\begin{bmatrix} M_1 & M_2 \end{bmatrix} = \begin{bmatrix} -0.7071 & 0.0000 & 0.7071 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 \\ 0.0000 & -1.0000 & 0.0000 & 0.0000 \\ -0.7071 & 0.0000 & -0.7071 & 0.0000 \end{bmatrix}\) found in step 1.1 leads to: \(A_{22} = 1.0000 \quad A_{12} = \begin{bmatrix} -0.7071 \\ 0.0000 \\ 0.0000 \\ 0.7071 \end{bmatrix} \).

A solution for step 1.2 is:

\[ S_{22} = 333.3318, \quad S_{21} = \begin{bmatrix} 0.0000 & 0.0000 & -2080.878778 \end{bmatrix} \]
which results in step 1.3:

\[ T = \begin{bmatrix} -1471.4028 & 333.3318 & 0.0000 & 1471.4028 \end{bmatrix}. \]

This solution has no invariant zeros and imply in step 2.1:

\[ Y = \begin{bmatrix} 0.6982 & 0.0000 & 0.1582 \\ -0.0650 & 0.0000 & 0.9853 \\ 0.0000 & 1.0000 & 0.0000 \\ 0.7129 & 0.0000 & -0.0650 \end{bmatrix}. \]

A solution for the step 2.2 is, then, found:

\[ Y = \begin{bmatrix} 10887.2550 & 150744.2000 & 304.1292 \\ -4945.7042 & 52069.6190 & -336.9173 \\ -16877.4870 & -336.9173 & 52208.9740 \end{bmatrix}. \]

The corresponding output feedback matrix that stabilizes the closed loop system is:

\[ G = \begin{bmatrix} -218.6662 & 348.0591 & 303.7491 \\ 52.7244 & -78.8494 & -75.4133 \end{bmatrix} \]

resulting:

\[ \sigma(A + BG) = \{ -3.4142 , -5.7409 , -6.2069 \pm 4.6210j \} \subset D_V. \]

Where the eigenvalue \( -3.4142 \subset D_T \) corresponds to step 1.

\[ \triangle \]

5. CONCLUDING REMARKS

In this paper, an output feedback approach and an algorithm were presented to compute solutions by LMI techniques. The solution of Sylvester equations was related to the solution of Lyapunov equations that allowed to obtain pole placement in LMI regions. Thus, necessary and sufficient conditions were presented for the existence of the static output feedback matrix. This paper also was proposed a numerically efficient solution algorithm for the coupled Lyapunov equations to determine the output feedback matrix.

REFERENCES


