CONTROL MODELS FOR CONSTRAINED MECHANICAL SYSTEMS – MODELING FOR CONTROL DESIGN REVISED

Elżbieta Jarzębowska

Warsaw University of Technology, Institute of Aeronautics and Applied Mechanics, Warsaw, Poland, elajarz@meil.pw.edu.pl

Abstract: The paper revises constraint types put upon mechanical systems and ways they are approached in control design. We demonstrate that a typical classification of constraints, modeling methods of analytical mechanics, which follow this classification, and constraints handling in control design, underexploit the role of effective constraints modeling. A unified control oriented approach to model constrained systems is presented.

Keywords: constrained systems, control oriented modeling.

1. INTRODUCTION

Constrained mechanical systems are an active research topic in both analytical mechanics and control. Constraints may be classified as nonholonomic and holonomic. We do not address holonomic system dynamics and control, since these topics can be considered solved problems, at least theoretically [1,2,3]. An elegant apparatus for modeling mechanical systems with first order nonholonomic constraints is generated using either variational or geometric mechanics, see e.g. [4,5,6]. Nonlinear control theory takes advantage of both formulations. If constraints are not first order, these approaches fail. It does not mean that there are no other constraint types on system motions. They are encountered very often, for example in robotic systems design, operation, safety, and performance specifications. The lack of modeling methods for systems with some general type of constraints caused that there was no attempt to describe task-based or control-based constraints by equations. An exception is a position constraint - trajectory to track, which is always presented in the equation form but is not merged into a constrained dynamic model [1,7]. Such non-material constraints are satisfied by actions of appropriately designed control algorithms. They are taken into account when a controller is designed; see for example a constraint on the trajectory curvature to make this trajectory available for a car-like vehicle [8-10].

The paper revises constraint formulations and ways they enter a controller design process. Constrained system control models either kinematic or dynamic are the basis for controller designs. Based on examples of constrained systems, the paper demonstrates that a typical constraint classification known form classical analytical mechanics is unsatisfactory to model constrained systems and design controllers effectively. Also, relying on classical mechanics we may show that dynamic models of some classes of constrained systems are of the same type, e.g. they can be derived using the Lagrange approach, but their dynamic control models are different and different control laws have to be pursued. We also demonstrate in the paper that modeling is not only a starting point in a control design process. Its role may be more significant in order to design a controller. Usually, the model building part of the control design process is treated as a routine activity that consists in deriving equations of the Lagrange type. Other constraints on a system, if there are any, are dealt with later on.

Control theory has had a fruitful association with analytical mechanics from its birth. This relationship was confirmed during the past decades with the emergence of a geometric theory for nonlinear control systems closely linked to the modern geometric formulation of analytical mechanics. The shared evolution of these fields reflected the needs to solve practical problems. In model-based control system analysis and design explicit dynamic equations of motion are required. Hence, a unified theoretical framework, including systematic generation of equations of motion for constrained systems is needed. It is vital then to revise the constrained systems modeling and their handling for control applications.

In the paper we present a more general way of including all the constraint equations, either holonomic or nonholonomic, into the system kinematics or dynamics at the level of modeling. It enables adopting existing control algorithms, even these dedicated to holonomic systems, without any major modifications. Any systematic survey of modeling methods is provided in the paper; this can be found in [5,6,11-18] and many others. Instead, the paper focuses on constraint specifications and on ways they are managed in mechanics and control settings. Specifically, we consider systems with constraints that can be of high order and of non-material type, like programmed constraints. For example, requirements on robots and their performance may be specified by constraint equations other than holonomic or first order nonholonomic [8-10,19]. Thus, in the paper we go beyond the scope of classical analytical mechanics.

The paper is organized as follows. In Sections 2 and 3 methods of modeling constrained systems are reviewed from the point of view of mechanics and control. In Section 4 examples that demonstrate a variety of constraints put on system motions are presented. In Section 5 a unification of modeling constrained systems is presented. The paper closes with conclusions and a list of references.
2. MODELING CONSTRAINED SYSTEMS IN MECHANICS

The history of mechanics of constrained, especially nonholonomic, systems is extraordinarily rich, e.g. [5,6,11-18]. At the beginning of the 20-th century it was observed that for some systems, like electro-mechanical systems, equations of motion do not have the form of Lagrange’s equations [20]. A new trend to leave Lagrange’s approach started. It was accompanied by efforts to eliminate unknown constraint reaction forces from equations of motion. The paper pays special attention to results obtained within this trend by Nielsen, Tzenoff, Mangeron and Deleanu [21-23]. Their work motivated the author to develop new results, which can be applied to control [24-27]. Methods of classical analytical mechanics are applied to systems with nonholonomic constraints of first order and Appell’s classical analytical mechanics are applied to systems with second order constraints. Nielsen’s equations can be derived from Lagrange’s equations or from the Jourdain principle [21]. They are for systems with first order nonholonomic constraints \( \varphi_\beta(1, q_\alpha, \dot{q}_\alpha) = 0, \) where \( \beta = 1, \ldots, k, \sigma = 1, \ldots, n, \) and have the form:

\[
\frac{\partial T}{\partial \dot{q}_\mu} \frac{\partial}{\partial q_\mu} - \frac{1}{2} \frac{\partial T}{\partial q_\mu} = Q_\mu, \quad \mu = 1, \ldots, m, m = n - k
\]  

(2.1)

where \( T \) is a system kinetic energy and \( Q_\mu \) are components of a \( m \)-dimensional vector of external forces, both in generalized coordinates represented by a \( n \)-dimensional vector \( q \). Equations of motion (2.1) and of constraints are a set of \( n \) equations for \( n \) unknown \( q_\beta \)’s.

Tzenoff derived new equations from the Gauss principle. They are referred to as Tzenoff’s equations of the second kind and have the form [22]

\[
\frac{1}{2} \left( \frac{\partial}{\partial q_\mu} \frac{\partial}{\partial \dot{q}_\mu} - \frac{\partial T}{\partial q_\mu} \right) = Q_\mu.
\]  

(2.2)

They are applied to systems with second order constraints. Other equations - Tzenoff’s equations of the third kind were derived from the postulated variational principle in the form

\[
\sum_{i=1}^{N} \left( F_{i} - m_{i} \ddot{r}_{i} \right) \delta \ddot{r}_{i} = 0,
\]

(2.3)

where \( F_{i} \) are components of a \( N \)-dimensional vector of external forces, \( m_{i} \) denotes mass of \( i \)-th particle, and \( r_{i} \) are components of a \( N \)-dimensional position vector such that \( \delta \ddot{r}_{i} \) are defined as:

\[
\delta \ddot{r}_{i} = \sum_{\sigma=1}^{n} \delta \ddot{q}_{\sigma} \ddot{q}_{\sigma} = \sum_{\sigma=1}^{n} \delta \ddot{q}_{\sigma} \ddot{q}_{\sigma} \quad \nu = 1, \ldots, N
\]

(2.4)

Tzenoff’s equations of the third kind have the form:

\[
\frac{1}{3} \left( \frac{\partial}{\partial q_\mu} \frac{\partial}{\partial \dot{q}_\mu} - \frac{\partial T}{\partial q_\mu} \right) = Q_\mu, \quad \mu = 1, \ldots, m
\]  

(2.5)

They can be applied to systems with third order constraints. The author does not know any continuation of this research on the derivation of equations of motion for systems with constraints of order higher than three until 1960-4h. For new equations (2.1), (2.2) and (2.5) virtual displacement concept has to be redefined compared with its formulation in classical mechanics [5,6].

The virtual displacement in the Appell-Chetaev meaning is an extension of the classical virtual displacement [5,6,18]. It is introduced for nonholonomic first order equations of constraints \( \varphi_\beta(1, q_\alpha, \dot{q}_\alpha) = 0, \) \( \beta = 1, \ldots, k, k < n \) that are nonlinear in \( \dot{q}_\alpha \). It satisfies the relation:

\[
\sum_{\alpha=1}^{n} \frac{\partial \varphi_\beta}{\partial q_\alpha} \delta \dot{q}_\alpha = 0.
\]

(2.6)

The virtual displacement (2.6) includes the definition of the virtual displacement for holonomic and first order linear nonholonomic constraints. It holds for ideal constraints.

A generalized virtual displacement is an extension of the virtual displacement in the Appell-Chetaev meaning [23]. For ideal \( p \)-th order constraint equations in the form

\[
G_\beta(1, q_\alpha, \dot{q}_\alpha, \ldots, \dot{q}_\alpha^{(p)}) = 0, \quad \beta = 1, \ldots, k
\]

(2.7)

Mangeron and Deleanu postulated the generalized virtual displacement that satisfies the relation

\[
\sum_{\alpha=1}^{n} \frac{\partial G_\beta}{\partial q_\alpha} \delta \dot{q}_\alpha = 0.
\]

(2.8)

The existence of three variational principles in classical mechanics, i.e. the d’Alembert, Jourdain and Gauss principles raises the question about their equivalence from the point of view of the generation of equations of motion. For holonomic systems these three principles are equivalent. For nonholonomic systems, they are equivalent for systems with constraint equations linear in velocities; see [5,6]. When nonholonomic constraint equations are nonlinear in velocities, it must be stated what is the virtual displacement for them and whether they are ideal. Systems for which virtual displacements are related to first order nonholonomic constraints by (2.6) are referred to as Appell-Chetaev systems and the d’Alembert, Jourdain and Gauss principles are equivalent for them [6]. In general, for systems that are not Appell-Chetaev these principles are not equivalent. In the paper we consider Appell-Chetaev systems that satisfy the generalized virtual displacement (2.8). A detailed discussion of variational principles can be found in [6,18].

A generalized variational principle for ideal constraints was postulated by Mangeron and Deleanu in the form

\[
\sum_{i=1}^{N} \left( F_{i} - m_{i} \ddot{r}_{i} \right) v_{i} = 0.
\]

(2.9)

For Appell-Chetaev systems, for \( p = 1, 2, 3 \) the principle (2.9) coincides with the classical mechanics principles and with the principle (2.3). The proof and comments on the principle can be found in [23,27].

To the best of the author’s knowledge, the principle (2.9) and equations obtained from it in a vector form in [23], were the limits of analytical mechanics results for the generation of equations of motion for constrained systems.

3. DYNAMIC CONTROL MODELS

Nonholonomic system models that include control inputs yield their control form, which can be presented as [3,4,28]

\[
\dot{x} = f(x,u),
\]

(3.1)

where \( x \in \mathcal{M} \), and \( \mathcal{M} \) is a smooth \( n \)-dimensional manifold referred to as the state space, \( u(t) \) is a time-dependent map from the nonnegative reals \( R^+ \) to a constraint set \( \Sigma \subset \mathcal{R}^n, f \)
is assumed to be \( C^\infty \) (smooth) or \( C^\alpha \) (analytic) and is taken from \( M \times \mathbb{R}^m \) into \( TM \) such that for each fixed \( u, f \) is a vector field on \( M \). The map \( u \) is assumed to be piecewise smooth or piecewise analytic, i.e. it is admissible. There are many generalizations and specializations of this definition, e.g. for Hamiltonian and Lagrangian control systems [4]. We consider system models popular in applications, which are affine nonlinear control systems, i.e.

\[
\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i, \quad (3.2)
\]

where \( f \) is the drift vector field and \( g_i, i=1,\ldots,m, m=n-k \), are the control vector fields, and both are smooth vector fields on \( M \). We assume that the constraint set \( \Sigma \subset \mathbb{R}^m \) contains an open neighborhood of the origin in \( \mathbb{R}^m \).

One significant feature of a nonholonomic control system is the number of inputs \( u_i(t) \). When it is equal to the number of degrees of freedom, a system id fully actuated. When it is less, a system is underactuated and it is regarded as a control system with second order nonholonomic constraints.

From the control theory perspective, nonholonomic constraints and their sources are viewed in a different way than in analytical mechanics. Usually, two situations are listed when nonholonomic constraints arise:

1. Explicit kinematic constraints.
2. Dynamic constraints preserved by Lagrange’s or Hamilton’s equations.

Underactuated systems are usually treated separately and classified as systems with constraints on controls, see [27] and references there.

The first conclusion based on this classification is that, excluding underactuated systems, nonholonomic constraints that control theory incorporates into its dynamics are of first orders and of the material type or they specify the conservation laws. The control theory approach to constraints can be illustrated by an example of a car-parking maneuver between two cars. If to ignore constraints, a path from a given initial position in a street to a final position between two cars would consist of two line segments, which is not feasible for a car. It is convenient then, to convert the problem with nonholonomic constraints into a steering problem. It means that we are not interested in directions a system cannot move, but in directions it can. Thus, instead of the constraint form \( B_i(q) \dot{q}_i = 0 \), where \( B_i \) is a \((k \times n)\) coordinate dependent matrix, which is true for most material kinematic constraint equations, we choose a basis for the right null space of the constraints, denoted by \( g_i(q) \in \mathbb{R}^n, i=1,\ldots,n-k \), where \( k \) is the number of constraint equations. The control problem for a nonholonomic system can be restated as finding input functions \( u_i(t) \in \mathbb{R}^{n-k} \) such that constraints on a system are presented as

\[
\dot{q} = g_i(q) u_i + \ldots + g_{n-k}(q) u_{n-k}, \quad (3.3)
\]

and the system can be driven from a given \( q_0 \) to \( q_1 \). It can be shown that if the matrix \( B_i \) elements are smooth and linearly independent, so are the \( g_i \)'s [2,4]. Most constraints listed in nonlinear control can be presented in the form (3.3), which is a driftless version of (3.2).

Nonholonomic constraint equations can be written as

\[
\dot{q}_i = \sum_{j=1}^{n} \phi_{ij}(q_1,\ldots,q_n) \dot{q}_j, \quad (3.7)
\]

where the first \( m \) out of \( n \) generalized velocities, in the number equal to the number of system’s degrees of freedom, are regarded as independent, \((n-m)\) velocities are dependent, and \( \phi_{ij}(q_1,\ldots,q_n) \) are smooth functions of their arguments. The generalized velocity vector can be partitioned in this way at least locally to obtain (3.7).

A specific class of nonholonomic constraints is called Chaplygin [29]. If constraint functions satisfy certain symmetry properties, namely that they are cyclic in the last \((n-m)\) generalized coordinates, we obtain Chaplygin nonholonomic constraint equations

\[
\dot{q}_i = \sum_{j=1}^{n} \phi_{ij}(q_1,\ldots,q_n) \dot{q}_j, \quad i = m+1,\ldots,n \quad (3.8)
\]

Equations (3.7) and (3.8) can be presented in the control form (3.2) by viewing the independent generalized velocities as inputs. Equations (3.7) take the form

\[
\dot{q}_i = \sum_{j=1}^{n} \phi_{ij}(q_1,\ldots,q_n) u_j, \quad i = m+1,\ldots,n \quad (3.9)
\]

If the Chaplygin assumption holds, equations (3.9) can be presented in the nonlinear control form as

\[
\dot{q}_i = \sum_{j=1}^{n} \phi_{ij}(q_1,\ldots,q_n) u_j, \quad i = m+1,\ldots,n \quad (3.10)
\]

Most systems with material nonholonomic constraints are Chaplygin and for this reason the kinematic control model (3.10) is a focus of many theoretic control studies, e.g. [4]. It can be presented in the so-called chained or power forms. Since these forms are used to model nonholonomic systems of practical importance such as wheeled vehicles, multibody spacecraft, or tractors with trailers, it is no surprise that many studies are focused on these classes of systems.

The nonholonomic constraint equations (3.7) or, equivalently, (3.9) that cannot be written in the form (3.8), or equivalently (3.10), are referred to as non-Chaplygin nonholonomic systems. In [30] an example of a kinematic control model of a non-Chaplygin is studied. It has to be pointed out that constraint equations which have been investigated so far were mostly in the so-called Chaplygin forms, mostly driftless and differentially flat, and could be transformed into the power or chained forms or to their extensions. A tracking control design for such systems can be considered a solved problem, at least theoretically [2,4].
Equations (2.1), (2.2) or (2.5) and their generalizations have not been used for any control applications. Nonlinear control uses dynamic models based upon Lagrange’s equations with multipliers, or their modifications, which are

\[ M(q)\ddot{q} + C(q, \dot{q}) + D(q) = J^T(q)\lambda + Q(q, \dot{q}), \]

\[ J(q)\dot{q} = 0, \]  
(3.11)

where the \((n \times n)\) matrix function \(M(q)\) is assumed to be symmetric and positive definite, \(C(q, \dot{q})\) is a \(n\)-vector function, \(D(q)\) is a vector of gravitational forces, \(Q(q, \dot{q})\) is a vector of external forces, and \(J(q)\) denotes a \((k \times n)\) matrix function, which is assumed to have full rank. All these functions are assumed to be smooth and defined on an appropriate open subset of the \((q, \dot{q})\) phase space. The scope of applications of these equations is significant, since only first order constraints, material and generated by conservation laws are merged into this dynamic model. Requirements for motion that are often specified by equations for controlled systems are not merged into (3.11).

First order constraints that are adjoined to the equations of motion via the introduction of Lagrange multipliers can also be embedded through the reduction procedure, which avoids the addition of auxiliary variables. Dynamic control models actually used, which we refer to as classical dynamic control models, are based on (3.11) and have the form [2,4]

\[ M(q)\ddot{q} + C(q, \dot{q}) + D(q) = J^T(q)\lambda + E(q)\tau, \]

\[ J(q)\dot{q} = 0, \]  
(3.12)

where \(\tau\) is a \(r\)-dimensional vector of control inputs. Transition from constrained dynamics (3.11) to control dynamics (3.12) requires some preprocessing work to eliminate constraint reaction forces. This transformation to the reduced-state form can be accomplished starting from Lagrange’s equations (3.11) written as

\[ \frac{d}{dt}\left(\frac{\partial T(q, \dot{q})}{\partial q}\right) - \frac{\partial T(q, \dot{q})}{\partial \dot{q}} = J^T(q)\lambda + Q(q, \dot{q}), \]

\[ J(q)\dot{q} = 0, \]  
(3.13)

where \(Q(q, \dot{q})\) stands for all applied external forces.

To eliminate constraint forces from (3.13) we project them onto the linear subspace generated by the null space of \(J(q)\). Since \((J^T(q)\lambda)\cdot\delta\dot{q} = 0\) Lagrange’s equations become

\[ \left[ \frac{d}{dt}\left(\frac{\partial T(q, \dot{q})}{\partial \dot{q}}\right) - \frac{\partial T(q, \dot{q})}{\partial q}\right]_{\delta\dot{q}} = 0, \]  
(3.14)

where \(\delta\dot{q} \in \mathbb{R}^n\) and \(J(q)\delta\dot{q} = 0\). We partition the vector \(q\) and the \(J(q)\) matrix such that \(q = (q_1,q_2) \in \mathbb{R}^{n-k} \times \mathbb{R}^k\), and \(J = [J_1(q) \ J_2(q)]\). \(J_2(q) \in \mathbb{R}^{n-k}^k\) is invertible. Then \(\delta\dot{q}_2 = -J_2^{-1}(q)J_1(q)\delta\dot{q}_1\). Inserting \(\delta\dot{q}_2\) to (3.14) we obtain

\[ \left[ \frac{d}{dt}\left(\frac{\partial T(q, \dot{q})}{\partial \dot{q}_1}\right) - \frac{\partial T(q, \dot{q})}{\partial q}\right]_1 - J_2^T \left[ \frac{d}{dt}\left(\frac{\partial T(q, \dot{q})}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial \dot{q}_2} \right]_2 = 0. \]  
(3.15)

Equations (3.15) are second order differential equations in terms of \(q\). They can be simplified by reusing the constraint equation \(\dot{q}_2 = -J_2^{-1}(q)J_1(q)\dot{q}_1\) to eliminate \(\dot{q}_2\) and \(\ddot{q}_2\).

The evolution of \(q_2\) can be retrieved by reapplication of the constraint equations. We can demonstrate that equations (3.15) with the constraint \(J(q)\dot{q} = 0\) are equivalent to Nielsen’s equations (2.1). It is enough to show that

\[ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) = \frac{\partial^2 T}{\partial q^2} \dot{q} + \frac{\partial^2 T}{\partial \dot{q} \partial q} \ddot{q} + \frac{\partial^2 T}{\partial \dot{q}^2} \dddot{q}, \]  
(3.16a)

\[ \dot{T} = \frac{\partial T}{\partial \dot{q}^2} \dddot{q}, \]  
(3.16b)

Based on (3.16b) and comparing (3.16a) and (3.17) we obtain

\[ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) = \frac{\partial T}{\partial \dot{q}^2} \dddot{q}, \]  
(3.18)

Relations (3.18) inserted into (3.15) and developed for each \(q_2\), \(\alpha = 1,\ldots,n\), yield that terms in brackets in (3.15) are equal to \(\frac{\partial T}{\partial \dot{q}^2} \dddot{q}\) so (3.15) are equivalent to (2.1).

The dynamic control model (3.12) can be transformed to the state space form obtained by an extension of the kinematic control model (3.2) or (3.10) as

\[ \dot{q} = g(q)\nu + g_v(q)\nu + g_{n-k}(q)\nu_{n-k}, \]  
(3.19a)

\[ v_i = u_i, \]  
(3.19b)

where \(r_i, \ldots, r_m\) denote an order of time differentiation and \(\nu\) is the output of a linear system consisting of chains of integrators. This reduction procedure is proposed by Campion et al. in [31]. Equations (3.19a,b) form a dynamic control model since in mechanics \(r_i = 1, i = 1,\ldots,n-k\), controls are typically generalized forces and the model consists of the constraint equation (3.19a) and the equations of motion (3.19b), which reduce to \(\nu = u\). To demonstrate this, consider the dynamics (3.12). Equation \(J(q)\dot{q} = 0\) constrains the velocity \(\dot{q}\) at each \(q\) to the null space of \(J(q)\). Let the vector fields \(g_1,\ldots,g_m, m = n-k\), form the basis for the null space of \(J(q)\) at each \(q\), and let \(g(q) = (g_1(q),\ldots,g_m(q))\). Then \(J(q)g(q) = 0\) for each \(q\) and the second of equations (3.12) can be presented as (3.19a) for some appropriately defined \(m\)-dimensional vector \(v = (v_1,\ldots,v_m)\). Components of \(v\) may or may not have physical interpretations as velocities. By differentiating (3.19a) we obtain

\[ \ddot{q} = g(q)\nu + \dot{g}(q)\nu. \]

Substituting the above into the first of equations (3.12), and premultiplying by \(g^T(q)\) we obtain

\[ g^T(q)M(q)g(q)\nu + F(q,\dot{q}) = g^T(q)E(q)\tau, \]

in which \(F(q,\dot{q}) = g^T(q)M(q)\dot{g}(q)\nu + C(q,\dot{q}) + D(q)\dot{q}\).

We assume that the map \(g^T(q)E(q)\) is onto what means that we require that independent degrees of freedom of the system are actuated. Then we can apply feedback.
linearization $U(\dot{q}, q, u): R^n \times R^n \times R^n \rightarrow R^n$ such that $\dot{v} = u$
where $u = (u_1, ..., u_m)$ is a $m$-dimensional vector control. In this way we get (3.19b) with $r_1 = 1$.

4. EXAMPLES OF CONSTRAINT FORMULATIONS FOR MECHANICAL SYSTEMS

Examples that follow demonstrate constraints of various sources that may be put on mechanical systems. It can be observed that for some systems kinematic and dynamic control models may be developed and for others only dynamic control models are available. Also, it is shown that a holonomic system subjected to a non-material constraint may become nonholonomic.

Example 1 -- two-wheeled mobile platform [27]
A two-wheeled mobile platform presented in figure 4.1 is a nonholonomic system with powered wheels.

![Fig. 4.1. Model of a two-wheeled mobile platform](image)

Its motion may be specified by a coordinate vector $q \in R^5$, where $q = (q_1, q_2, q_3, q_4, q_5)$, $q_1 \in R^2$, $q_2 \in R^2$ and $q_3 = (x, y, \phi)$, $q_4 = (\varphi, \dot{\varphi})$. Motors control the two wheels independently. Angles $\phi, \varphi, \psi$ denote the heading angle, wheel angles due to rolling for the right and left wheel, respectively. The distance between the wheels is equal to $2b$ and the robot mass center $C$ is located at a distance $d$ from the geometric center $O$. Material constraint equations for the mobile robot specify the condition that it does not slip sideways and its driving wheels do not slip. They have the form

$$\begin{align}
\dot{x}_C \cos \phi - x_C \sin \phi \dot{\phi} d &= 0, \\
\dot{x}_C \cos \psi + \dot{y}_C \sin \psi \dot{b} &= \dot{\phi} r, \\
\end{align}$$

where $\dot{x}_C, \dot{y}_C$ - components of the velocity of the mass center $C$, and $c_1$ is a constant of integration. Angles $\phi, \varphi, \psi$ can be defined in such a way that $c_1$ may be taken to be zero. The constraints (4.1) consist of two nonholonomic and one holonomic constraint. The number of coordinates is $n=5$, the number of the constraint equations is $k=3$, the number of degrees of freedom is $n-k=2$, and the number of control inputs is 2. Then, angular velocities of the wheels or control torques $\tau_\varphi, \tau_\psi$ may be the inputs. The mobile platform may be controlled at a kinematic or dynamic level. The same is true for other systems with powered wheels such as cars, scooters or bikes. However, for heavy wheeled vehicles, operating at high speeds, satisfactory control results can be achieved by model-based control strategies [32].

Example 2 -- snake-board [33]
The snake-board presented in figure 4.2 is a nonholonomic system with passive wheels. A rider has to use body cyclic motions and muscles of legs and hands to ride the vehicle. By coupling a twisting of the human torso with an appropriate turning of the wheel assemblies (the passive wheel assemblies can pivot freely about a vertical axis), where the turning is controlled by the rider’s foot movements, the rider can generate a snake-like locomotion pattern without having a kick off the ground.

![Fig. 4.2. Model of a snake-board](image)

Motion of the snake-board may be specified by a coordinate vector $q \in R^5$, $q = (x, y, \theta, \psi, \varphi)$, where $(x, y, \theta)$ are position variables and $(\psi, \varphi)$ - shape variables.

Equations of nonholonomic constraints for the snake-board that come from rolling its wheels without slipping are

$$\begin{align}
-x_\psi \sin(\theta + \varphi) + y_\psi \cos(\theta + \varphi) - \theta \cos \varphi &= 0, \\
-x_\psi \sin(\theta - \varphi) + y_\psi \cos(\theta - \varphi) + \theta \cos \varphi &= 0.
\end{align}$$

The number of coordinates is $n=5$, the number of the constraint equations is $k=2$, the number of degrees of freedom is $n-k=3$, and the number of control inputs is 2. There may be control torques $\tau_\psi, \tau_\varphi$ that represent the twisting of the human torso and the rider’s foot movements. Control at the dynamic level is needed for the snake-board since there are no control variables in the system kinematics.

Example 3 -- roller-racer [34]
The roller-racer motion is very different from that of the snake-board, since its rider does not have to move his body. The propulsion and steering of the roller-racer come from a rotary motion at the joint that connects its segments. The joint torque is the only control of the system. Motion of the roller-racer is usually specified by a coordinate vector $q = (x, y, \theta, \psi)$, where $(x, y, \theta)$ are position variables and $\psi$ is a shape variable.
Equations of nonholonomic constraints are
\[ x \sin \theta - y \cos \theta = 0, \]
\[ -\dot{x} \sin \psi + y \cos \psi + \dot{\theta} (\theta - \psi) + \dot{l} y = 0. \]  (4.3)
The number of coordinates is \( n=4 \), the number of the constraint equations is \( k=2 \), the number of degrees of freedom is \( n-k=2 \), and the number of control inputs is \( I \). The only control torque is \( \tau \) that controls the rotary motion at the joint. The purely kinematic controller design for the roller-racer is not possible. It may be controlled at the dynamic level only.

Example 4 – snake-like robot [35]
The snake-like robot is a longer snake than the roller-racer.

The robot may locomote by bending its body using the side force against side slip. The nonholonomic constraint equations that come from rolling its wheels without slipping have the same form as for other wheeled vehicles, i.e. \( A(q)\dot{q} = 0 \), where \( A \) is a \((n+2 \times n+2)\) matrix. There is one nonholonomic constraint equation for each segment so the snake-like robot may be controlled at a dynamic level only.

Example 5 – two-link planar manipulator [27]
A planar two-link planar manipulator model is presented in figure 4.5. It moves in the horizontal plane \((x,y)\). Two degrees of freedom are described by \( \Theta_1, \Theta_2 \).

It is a holonomic system, however, we may formulate a requirement that the manipulator end-effector is to move along a trajectory for which its curvature changes according to a specified function \( \Phi^* \). In the task-space \((x,y)\) this constraint has the form
\[ \ddot{x} = -\Phi^*(x^2 + y^2)^{3/2} \frac{3(\dddot{x}y - \dddot{y}x)}{y^3} + \frac{\dddot{y}x}{y^2}. \]  (4.6)
In joint coordinates \((\Theta_1, \Theta_2)\), by inserting \( x = l \cos \Theta_1 + l \cos (\Theta_1 + \Theta_2), y = l \sin \Theta_1 + l \sin (\Theta_1 + \Theta_2) \) and their time derivatives into (4.6), we obtain
\[ F_2 \ddot{\Theta}_1 + \ddot{\Theta}_2 - F_1 = 0, \]  (4.7)
where
\[ F_1 = \frac{A_2 - A_1}{a_2 + a_3 a_5}, \quad F_2 = \frac{a_1 + a_2 + a_4 (a_3 + a_5)}{a_2 + a_3 a_6}, \]
\[ A_2 = -\Phi^*(a_3^2 + a_4^2)(a_3^2 + a_5^2) + 3\Phi^*(a_3 a_5 + a_4 a_6), \]
\[ A_1 = 3a_3 (\ddot{\Theta}_1 + \ddot{\Theta}_2) + 3a_4 (\ddot{\Theta}_1 + \ddot{\Theta}_2) (\ddot{\Theta}_1 + \ddot{\Theta}_2) - a_2 a_5 (\dot{\Theta}_1 + \dot{\Theta}_2)^2, \]
\[ A_2 = 3a_3 (\ddot{\Theta}_1 + \ddot{\Theta}_2) + 3a_4 (\ddot{\Theta}_1 + \ddot{\Theta}_2) (\ddot{\Theta}_1 + \ddot{\Theta}_2) + a_4 a_5 (\dot{\Theta}_1 + \dot{\Theta}_2)^2, \]
\[ \alpha_1 = -l \sin \Theta_1, \quad \alpha_2 = l \sin \Theta_1, \quad \alpha_3 = a_1 a_4 a_6, \]
\[ \alpha_4 = -a_2 a_4 a_6, \quad \alpha_5 = -a_3 a_4 a_6, \quad \alpha_6 = a_1 a_4 a_6, \]
\[ \alpha_7 = a_1 (\ddot{\Theta}_1 + \ddot{\Theta}_2) + a_4 (\dot{\Theta}_1 + \dot{\Theta}_2) + a_5 (\dot{\Theta}_1 + \dot{\Theta}_2)^2, \]
\[ \alpha_8 = -a_3 a_4 a_6, \quad \alpha_9 = a_1 a_4 a_6 + a_2 a_4 a_6 + a_3 a_4 a_6. \]
Equation (4.7) is a nonholonomic constraint equation of the third order and it is task-based. It can be thought of as a task that specifies writing on a surface, painting or scribing. The Lagrange approach fails for motion description of the manipulator constrained by (4.7).
Example 6 – free-floating space manipulator [27]
A space two-link manipulator shown in figure 4.6 consists of a base described by its moment of inertia $J$ and the orientation angle $\phi$ relative to a fixed axis in the plane. Let $\theta_1$ be the angle of the first link of mass $m_1$ and length $l_1$ relative to the base, and $\theta_2$ be the angle of the second link of mass $m_2$ and length $l_2$ relative to the first one.

![Fig. 4.6. Model of a two-link space manipulator](image)

For simplicity we assume that link masses are concentrated at the ends of the links. In our model the manipulator base is pinned to the ground at its center. Pinning the base permits the body to rotate freely but prevents translation. Free-floating systems are considered nonholonomic systems with first order constraints that come from the angular momentum conservation. When the angular momentum is zero, the nonholonomic constraint has the form

$$ f + \left[ J + \left( m_1 + m_2 \right) \dot{\theta}_2^2 + m_1 l_2^2 \right] \phi + \left[ m_1 + m_2 \right] \dot{\theta}_1 + m_2 l_2 \dot{\theta}_2 + m_1 l_1 \cos \phi \left( 2 \dot{\phi} + \dot{\theta}_1 + \dot{\theta}_2 \right) = 0. \quad (4.8) $$

The constraint equation (4.8) has the same form as the one that comes from the condition of rolling wheels without slipping, i.e. it is $A(q) \dot{q} = 0$.

Example 7 – planar model of a diver [36]
A model of a diver that performs a somersault is presented in figure 4.7. The configuration of a diver is described by $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$. The nonholonomic constraint equation origins from the conservation of the angular momentum. If the initial angular momentum is equal to $K_0$, then

$$ K_0 = \left[ a_1 + 2 b \cos \theta_1 + 2 c \sin \theta_2 + 2 d \cos \theta_2 + 2 e \sin \theta_3 + 2 f \cos \left( \theta_3 - \theta_2 \right) + 2 g \sin \left( \theta_3 - \theta_2 \right) \right] \dot{\theta}_1 + \left[ a_2 + b \cos \theta_1 + b \sin \theta_2 + d \cos \left( \theta_2 - \theta_3 \right) + b \sin \left( \theta_2 - \theta_3 \right) \right] \dot{\theta}_2 + \left[ a_3 + b \cos \theta_1 + b \sin \theta_3 + d \cos \left( \theta_3 - \theta_2 \right) + b \sin \left( \theta_3 - \theta_2 \right) \right] \dot{\theta}_3 \quad (4.9) $$

where $a_1, b, c, d, e, h, s, f$ are constant coefficients. Equation (4.9) may be written as

$$ K_0 = \left[ B_1(\theta_2, \theta_3) B_2(\theta_2, \theta_3) B_3(\theta_2, \theta_3) \right] \left[ \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3 \right]^T. $$

A kinematic control model for the diver can be obtained by setting angular velocities $u_1 = \dot{\theta}_1, u_2 = \dot{\theta}_2, u_3 = \dot{\theta}_3$ as control inputs. It yields

$$ \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} K_0 \\ -B_2 \\ -B_3 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} -B_1 \\ B_1 \\ 0 \end{bmatrix} u_2 = f(\theta_2, \theta_3) + g_1(\theta_2, \theta_3) u_1 + g_2(\theta_2, \theta_3) u_2. \quad (4.10) $$

A drift term $f(\theta_2, \theta_3)$, which is typical in biomechanical system models, appears and it is a nonlinear function of shape variables. It can be verified that the control model (4.10) is STLC. Indeed, a distribution spanned by the brackets $g_1, g_2$ and

$$ [g_1, g_2] = g_3 = \begin{bmatrix} 1 \\ \frac{B_1}{B_2} \frac{\partial B_3}{\partial \theta_2} - \frac{B_1}{B_3} \frac{\partial B_2}{\partial \theta_3} - \frac{B_1}{B_3} \frac{\partial B_1}{\partial \theta_3} + \frac{B_1}{B_3} \frac{\partial B_2}{\partial \theta_3} \\ 0 \\ 0 \end{bmatrix} $$

has dimension 3 beyond isolated singularities $(\theta_2, \theta_3)$ for which $g_3$ is equal to zero. The system may be controlled at the kinematic or dynamic level.

Example 8 - Grioli’s example [37]
According to Grioli’s theorem, the necessary and sufficient condition for a pseudo-regular precession motion for a rigid body is

$$ (p \dot{q} - q \dot{p}) + r(p^2 + q^2) - \lambda (p^2 + q^2)^{1/2} = 0, \quad (4.11) $$

where $\lambda = \text{const.}, p = \omega_x, q = \omega_y, r = \omega_z$, i.e. they are angular velocities of the body in the $(\xi, \eta, \zeta)$ frame fixed in the body. The rotating body is presented in figure 4.8.
Fig. 4.8. Pseudo-regular precession of a body

The quasi-velocities \( \omega_x, \omega_y, \omega_z \) may be presented using Euler’s angles \( \varphi, \psi, \theta \) as follows \([5,6]\)

\[
\omega_x = \dot{\psi} \sin \theta \cos \varphi + \dot{\theta} \sin \psi \cos \varphi,
\omega_y = \dot{\psi} \sin \theta \sin \varphi - \dot{\theta} \cos \psi \sin \varphi,
\omega_z = \dot{\psi} \cos \theta + \dot{\varphi}.
\]

Inserting (4.12) into the Grioli condition (4.11) we obtain

\[
\dot{\psi} \sin \theta \cos \varphi - \dot{\theta} \sin \psi \sin \varphi + 2 \dot{\psi} \cos \theta - \dot{\varphi} = 0.
\]

The constraint equation (4.13) is task-based and it is second order nonholonomic. Appel’s or Tzenoff’s equations may be applied to derive motion equations of the body subjected to the constraint (4.13).

Example 9 – tracking a moving target \([38,39]\)

In \([38,39]\) control and guidance of missiles is studied. A concept of a programmed motion appears there in the context of trajectory tracking, i.e. tracking a moving target. The problem is illustrated in figure 4.9.

![Fig. 4.9. Tracking a moving target](image)

It is formulated as follows: A target Q moves a prescribed motion \( \xi(t) \) along \( ax \). A follower, modeled as a particle, moves in the \((x,y)\) plane in such a way that its velocity vector is directed towards Q. The constraint equation for the follower is

\[
y \dot{x} + (\xi - x) \dot{y} = 0.
\]

The constraint (4.14) task-based and may be nonholonomic.

Example 10 – motion along a predefined trajectory \([19,25]\)

Consider again the mobile platform from fig. 4.1 and specify the task-based constraints, which may be a control goal, motion or design limitations, e.g. a desired trajectory equation

\[
B_t = x^2 + y^2 - R(t) = 0, \quad R(t) = 0.2 + 0.01t
\]

or a rate of change of a trajectory curvature

\[
\frac{\dot{x}}{y} = -\Phi(\dot{x}^2 + \dot{y}^2)^{3/2} - \lambda(\dot{x} \ddot{x} + \dot{y} \ddot{y}) + \frac{\dot{y}^2}{\dot{y}^2 + \dot{x}^2}
\]

Other constraints of the type (4.16) may model required motion properties. The set of constraints for the mobile platform consists then of (4.1) and (4.16), i.e. the material constraints are to be supplemented by the task-based. For the system constrained by (4.16) the Lagrange approach fails. In \([8-10]\) such constraints are incorporated into a control design process at the level of a controller design. It requires, however, a modification of a controller each time a different constraint is imposed upon a system motion.

Example 11 - modeling constraints \([40]\)

For modeling robotic systems, coordinates other than generalized may be useful. Consider a four-bar mechanism and model its kinematics in natural coordinates, see figure 4.10. They are Cartesian coordinates \((x_1,y_1,x_2,y_2)\).

![Fig. 4.10. Model of a four-bar mechanism using natural coordinates](image)

The mechanism consists of rigid links and it results in three position constraint equations in the form

\[
L_1(x_1-x_2)^2 + (y_1-y_2)^2 - L_1^2 = 0,
L_2(x_2-x_3)^2 + (y_2-y_3)^2 - L_2^2 = 0,
L_3(x_3-x_4)^2 + (y_3-y_4)^2 - L_3^2 = 0.
\]

The same mechanism may be modeled using relative coordinates, see figure 4.11.

![Fig. 4.11. Model of a four-bar mechanism using relative coordinates](image)

The relative coordinates produce no constraints for open chains. However, for the closed chain the constraint equations are as follows
From the examples above we may conclude that constrained, specifically nonholonomic, systems are a large class of systems. From the perspective of mechanics and derivation of equations of motion for them, most of them belong to two classes of systems subjected to holonomic or first order nonholonomic constraints. They may be approached by Lagrange’s equations with multipliers and these equations are used to generate dynamic control models for them most often. However, not all systems with task-based or control-based constraints may be approached by Lagrange’s equations since they fail for systems with constraints of order higher than one. Also, we could see that a holonomic system like a planar manipulator, may become nonholonomic by a constraint put upon its motion properties.

From the perspective of nonlinear control, the constrained systems presented in the examples above differ and may not be approached by the same control strategies and algorithms. Some of them may be controlled at the kinematic level and the other at the dynamic level only. Their control properties depend upon the way they are designed and propelled, i.e., how many control inputs are available or whether their wheels are powered or not. They are divided into two control groups, which are treated separately, i.e., fully actuated and underactuated systems.

Based on the roller-racer and four-bar mechanism examples, we could see that the selection of coordinates for modeling constraints to control may be significant.

We may conclude then, that from the nonlinear control perspective a system design, way of its propulsion, control goals, task-based or work-space constraints may determine the way of the control oriented modeling for a controller design.

5. MODELING CONSTRAINED SYSTEMS FOR CONTROL APPLICATIONS

In control oriented modeling there are more constraint sources. We show in this Section that they may be treated as non-material constraints on systems. From the control perspective constraint sources can be as [27]:

1. Material constraints.
2. Conservation laws.
3. Design constraints that may arise from bounded linear and angular velocities or accelerations, or from a bounded trajectory curvature for a wheeled vehicle.
4. Control constraints that may arise from the number of control inputs.
5. Programmed constraints that are task-based and may arise from specifications of tasks or other requirements put by a designer or a control engineer.

The idea is to develop a unified constraint formulation, which may include the constraint types listed above, and develop a unified dynamic model of a system subjected to such constraints. The unified constraint formulation is as follows [26,27]

\[
B(t, q, \dot{q}, ..., \ddot{q}^{(p-1)}) \dot{q}^{(p)} + s(t, q, \dot{q}, ..., \ddot{q}^{(p-1)}) = 0,
\]

where \( p \) is the constraint order, \( q \) is a \( n \)-vector of generalized coordinates, \( B \) is a full rank \( (k \times n) \) matrix with \( n > k \) and \( s \) is a \( k \)-vector. We assume that (5.1) are linear in \( p \)-th order derivative of coordinates or we can transform them to this form. They may specify both material and non-material constraints since the type of a constraint equation does not influence the generation of equations of motion of a system subjected to it. The only concern is a constraint order and whether it is ideal. For order \( p = 0 \) we get a position constraint, which may be material and specify for example a constant distance between link ends or be a programmed constraint on a trajectory for a system. When \( p = 1 \) a constraint equation may be material and specify a condition of rolling without slipping. Also, it may arise from the conservation law or be a programmed constraint on a desired velocity of a system. Material constraints are of orders \( p = 0 \) or \( p = 1 \), the equation of the conservation law is of order \( p = 1 \), and constraint equations for \( p > 1 \) are of the non-material type. For (5.1) we introduce a definition.

**Definition:** The equations of constraints (5.1) are completely nonholonomic if they cannot be integrated with respect to time, i.e., constraint equations of a lower order cannot be obtained.

We assume that (5.1) are completely nonholonomic. Then they do not restrict positions \( q(t) \) and their derivatives up to \((p-1)\)-th order. Our definition is an extension of a definition of completely nonholonomic first order constraints [4] and completely nonholonomic second order constraints [41]. Necessary and sufficient integrability conditions for differential equations of arbitrary order such as (5.1) are formulated in [42].

A unified dynamic model of a system with the constraints (5.1) is derived using the generalized programmed motion equations (GPME) according to the following algorithm [26,27]. This modeling framework does not depend upon the constraint source and its type and order.

**Algorithm**

Assume that (5.1) may be solved, at least locally, with respect to a vector \( q^{(p)}_m \) of dependent coordinates, i.e.,

\[
q^{(p)}_m = q^{(p)}_m(t, q, \dot{q}, ..., \ddot{q}^{(p-1)}_m)
\]

and \( q = (q_\beta, q_\mu), q_\beta \in R^k, q_\mu \in R^{n-k} \).

1. Construct a function \( P_p \) such that

\[
P_p = \frac{1}{2} \left[ T^{(p)} - (p + 1)T_0^{(p)} \right]
\]

and \( T \) is the kinetic energy of an unconstrained system, \( T^{(p)} \) is its \( p \)-th order time derivative, and \( T_0^{(p)} \) is defined by

\[
T_0^{(p)} = \sum_{\alpha} \frac{\partial T}{\partial q^{(p)}_\alpha}.
\]

2. Construct a function \( R_p \) such that
3. Construct a function $R_p$, in which $q^{(p)}$ from (5.4) are replaced by the constraints (5.1)

$$R_p = p - \sum_{s=1}^{n} Q_s q_s^{(p)} = R_p (t, q, \dot{q}, \ddot{q}_p, q^{(p)}, q^{(p+1)})$$  \hspace{1cm} (5.4)

4. Assuming that components of a vector of external forces satisfy $\mathcal{Q}_a / \mathcal{Q}_a^{(p)} = 0$, equations of the programmed motion for a system with the constraints (5.1) have the form

$$\frac{\partial R_p}{\partial q^{(p)}} + \sum_{s=1}^{n} \frac{\partial R_p}{\partial q_s^{(p)}} \frac{\partial q_s^{(p)}}{\partial q^{(p)}} = 0, \quad \mu = k + 1, \ldots, n$$  \hspace{1cm} (5.6)

Equations (5.6) and (5.1) admit the following properties.

**Property 1:** Equations (5.6) are (n-k) second order differential equations and together with (5.1) can be presented as [26]

$$M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + D(q) = \tau_p,$$  \hspace{1cm} (5.7)

where $M(q)$ is a (n-k)×n inertia matrix, $V(q, \dot{q})$ is a (n-k)-velocity dependent vector, $D(q)$ is a (n-k)-vector of gravity forces, and $Q(t, q)$ is a (n-k)-vector of external forces. Equations (5.7) are referred to as a unified dynamic model of a constrained system.

**Property 2:** Equations (5.7) are free of constraint reaction forces, which are eliminated in the derivation process, i.e. (5.7) are in the reduced-state form. It is then a control oriented dynamics model.

**Property 3:** Dynamic models of systems with the constraints of order $p=1$ derived by the Lagrange approach and transformed to the reduced-state form are peculiar cases of (5.7). Equations (5.7) can replace Lagrange’s, Nielsen or Tzenoff’s equations.

For control purposes we introduce definitions.

**Definition 1** [26]

The unified dynamic model of a constrained system (5.7) is referred to as a reference dynamic model for programmed motion, shortly the reference dynamics. It is the extension of the models reported in [43,44] which apply to holonomic and first order nonholonomic systems. The reference dynamics (5.7) serves a programmed motion planning. It is defined as follows.

**Definition 2** [26]

Programmed motion planning for a system subjected to the constraints (5.1) consists in finding time histories of positions $q_p(t)$ and their time derivatives in motion consistent with the constraints. Specifically, in this formulation trajectory planning consists in obtaining a solution $q_p(t)$ of (5.7), in which a programmed constraint equation is algebraic. Solutions of (5.7) also serve verification whether a programmed constraint is eligible for a system, i.e. whether the system is capable of reaching desired positions, velocities and accelerations needed to follow the program, and the programmed constraint does not violate other constraints.

Based on the GPME, a unified dynamic control model may be derived. Including control inputs into the GPME for $p=1$, the unified dynamic control model takes the form

$$M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + D(q) = \tau_p,$$  \hspace{1cm} (5.8)

Equations (5.8) consist of (n-k) equations of motion and k equations of material constrains and conservation laws only. The matrix $M(q)$ is then (n-k)×n and $B_1(q)$ is a full rank (k×n) matrix. Since the constraints in (5.8) are linear first order, $C(q, \dot{q}) \dot{q}$ quantifies effects of Coriolis and centripetal forces. We assume that external forces, which are not controls, can be added to the left-hand side of (5.8).

The following properties of (5.8) can be derived from properties 1-3.

**Property 4:** The unified dynamic control model (5.8) is equivalent to the reduced-state Lagrange equations. Then, it replaces the dynamic control model based on Lagrange’s approach that nonlinear control uses.

**Property 5:** The unified dynamic control model (5.8) can be presented in a standard control form by reusing of the constraint equations presented as $\ddot{q} = G(q, \dot{q})$, where partition of $q$ is $q = (q_1, q_2)$. Vectors $q_1 \in \mathbb{R}^{n-1}$ and $q_2 \in \mathbb{R}^k$ are vectors of independent and dependent coordinates, respectively. Columns of the matrix $G(q)$ span the right null space of $B_1(q)$. It is a (n×m) matrix, $m = n-k$, of the form

$$G = -B_1^T(q) B_1(q),$$

where $I$ is a (m×m) identity matrix, $B_1^T(q) B_1(q)$ is a locally smooth (k×m) matrix function. The matrix $B_1(q)$ is expressed as $B_1 = [B_{11}(q), B_{12}(q)]$, $B_{11}(q)$ is a k×(n-k) matrix function, and $B_{12}(q)$ is a (k×k) locally nonsingular matrix function. Elimination of second order derivatives of dependent coordinates from the first of equations (5.8) yields

$$\ddot{q} = G(q, \dot{q}),$$  \hspace{1cm} (5.9)

where:

- $\ddot{q} = M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + D(q)$,
- $\ddot{q} = G(q, \dot{q})$,
- $\ddot{q} = M(q) G(q) + C(q, \dot{q}) G(q)$.

Equations (5.9) are exactly the reduced-state Lagrange equations of a nonholonomic system; compare to [43,44].

**Property 6:** There exists a static state feedback $U(q, \dot{q}, u) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that (5.9) can be transformed to the state-space control form. To this end, introduce a new state variable vector $x = (q, \dot{q}) = (x_1, x_2)$, $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^n$, for which (5.9) takes the form

$$\ddot{x}_1 + C_r(x_1, \dot{x}_1) \dot{x}_1 + D_r(x_1) = \tau_p,$$  \hspace{1cm} (5.10)

A static state feedback $U(x_1, x_2, u) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined as $\ddot{x}_1 + C_r(x_1, \dot{x}_1) \dot{x}_1 + D_r(x_1) = \tau_p$ can be selected and applied to (5.9) yields

$$\ddot{x}_1 = G(x_1, x_2),$$  \hspace{1cm} (5.11)

which is a desirable state-space control form (3.1).
The two dynamic models, i.e. the reference dynamics (5.7) and the unified dynamic control model (5.8) are control oriented and can be directly employed to design a control strategy. The strategy for tracking motions specified by (5.1) is developed in [26,27]. It is referred to as the model reference tracking control strategy for programmed motion. Details about the strategy can be found in [26,27].

Advantages of the GPME control oriented modeling framework are as follows:
- It is not sensitive to the constraint order. This is in contrast to current modeling methods for constrained systems, which treat separately constraints of different orders.
- The reference dynamics (5.7) captures high order nonholonomic constraints on systems and enables planning any programmed motion, i.e. trajectory tracking may be extended to programmed motion tracking.
- The separation of the programmed constraints from others causes that the unified dynamic control model (5.8) is equivalent to models used in nonlinear control.
- The equivalence of (5.8) and models based on the Lagrange approach promotes adaptation of existing control algorithms even those dedicated to holonomic systems.
- It uses one dynamic control model (5.8) to both holonomic and nonholonomic systems.
- Underactuated systems may be treated in the same way as other constrained systems [26].
- A library of reference models that plan different tasks can be generated off-line and stored in a computer. They all can be applied to one unified dynamic control model (5.8).

To demonstrate the Algorithm for the GPME in action, consider the model of the two-wheeled mobile platform and put the programmed constraint (4.15) on it. The reference dynamics (5.7) has the form [26]

\[ m_1 \ddot{\phi}_1 + m_2 \ddot{\phi}_2 + m_3 (\dot{x} + m_4 \dot{\psi} + m_5 \dot{\phi} + v_1 = 0, \]
\[ y \cos \phi - x \sin \phi - d \dot{\phi} = 0, \]
\[ \ddot{x} \cos \phi + \dot{y} \sin \phi + b \dot{\phi} = r \phi_1, \]
\[ \ddot{x} \sin \phi - \dot{y} \cos \phi - b \dot{\phi} = r \phi_1, \]
\[ x^2 + y^2 - 0.2 + 0.01t = 0. \]

where \( m_i \) are coefficients of the equation of the reference dynamics. The reference motion pattern is shown in figure 5.1. The same mobile platform may be subjected to the constraint (4.16) and we select two functions of the curvature: \( \Phi_1 = 5 \) and \( \Phi_2 = 2 \sin t + 1 \). The reference dynamics for these different constraints changes. The dynamic control model remains the same due to the separation of material and programmed constraints and it can be derived by the GPME for \( p = 1 \). Outputs of the reference dynamics are plugged into the control dynamics and the same control algorithms may be used. In figures 5.1 and 5.2 a computed torque is used.

\[ \text{CONCLUSIONS} \]

In the paper we presented the general way of the specification of constraints put on system motions and the framework for their incorporation into the system kinematics or dynamics at the level of modeling. It results in control oriented models that enable adopting existing control algorithms, even those dedicated to holonomic systems, without any major modifications.

\[ \text{REFERENCES} \]
