QUENCHING IN A CLASS OF SINGULARLY PERTURBED MECHANICAL SYSTEMS

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Abstract: In this paper we get some results about the dynamics of a class of mechanical systems under strong dissipation. We show the existence of unstable hyperbolic periodic orbits as well as the existence of quenching for the system (24).

Keywords: Singularly Perturbed Systems, Quenching, Integral Manifolds

1. INTRODUCTION

This paper deals with a class of mechanical systems under very strong dissipation. An example for this is an elastic structure under wind forces. In these situations there is no way controlling some parameters. So it is more adequate to suppose that those ones are big. Hence, in most of the cases, the mathematical model leads to singular perturbation problems, see (40). And one the most effective ways to deal with them is to use Integral Manifolds, see Appendix. One of the advantages of this approach is that it leads to a dimension reduction. In addition, under the condition of stability of the Integral Manifold, see (41), the dynamics on the Integral Manifold holds for the original system. Moreover, on this manifold the problem reduces itself to a regular one.

In this paper a "singular" version of a problem given at [1] it has been taken into account, see (2). In fact we deal with a generalization of it, see (3). The main reason for this is that, after dimension reduction, the Regular Perturbation Theory can be used everywhere. The algebraic computations are huge, but nowadays this is not a genuine problem. In [2] the same problem was tackled, but there are two main differences. First, the Averaging Method is not used here. Second, in Section 5 a case of quenching is found out. In Section 2 a mechanical problem which has motivated our research is described and in Section 3 a generalization of it is given that includes the non-autonomous case. A reduction to the integral manifold, even in this general setting, is performed. In spite of this, examples on the non-autonomous will not be worked. This one deserves another paper. In Section 4 an autonomous case is investigated. In fact is proved, under some general conditions, that (23) has unstable hyperbolic periodic solutions. In Section 5 the most delicate case is investigated, the equilibrium point (0, 0, 0). The arguments given in this section are a blend of rigorous and informal ones. Despite this, an interesting conclusion can be obtained. In fact there is a quenching, in the sense of [3] and [4], at the system (24). Moreover, this dynamic effect depends only on the signal of b. This result matches with the numerical analysis performed in that section, see Figures 2, 3, 4, 5.

2. A MECHANICAL PROBLEM

Let us consider the following parametric mechanical system given in Figure 1.

The equations of motion of this system are the following ones

\[
\begin{align*}
\dot{x}'' + \delta x' x'' + \delta_0 x' + (\omega_1^2 + c \cos (2 \eta t)) x + \gamma x' & = 0, \\
\dot{y}'' + \kappa y' + \omega_2^2 y + b x^2 & = 0.
\end{align*}
\]

(1)

Figure 1 – Mechanical System with Four Springs under parametric excitation

It is assumed that the damping coefficients \(\delta, \delta_0, \kappa\) are non negative parameters as well as the amplitude \(c\) of the parametric excitation and the nonlinear coefficient \(\gamma\) of the nonlinear horizontal spring.
If one takes $c, \delta, \delta_0, a, b, \kappa$ small, this problem has been investigated in [1] using the averaging method. Our aim here is different. We are asking for the existence and stability of periodic orbits of (1) under very strong horizontal dissipation. In view of this is reasonable to assume $\delta = 0$ and $\kappa = 0$. In other words, take $\delta_0 = \mu$, $\delta = 0$ and $\kappa = 0$ at (1) then one has
\[
\begin{align*}
&x'' + \mu x' + (\omega_1^2 + c \cos (2 \eta t)) x + 
\gamma x^3 + a x y = 0, \\
y'' + \omega_2^2 y + b x^2 = 0,
\end{align*}
\]
where $\mu \gg 1$. However we have worked in a much more general class of problems than (2), see Section 3. Anyway (2) was our motivating example.

3. A GENERAL FORMULATION

Consider the following system
\[
\begin{align*}
x'' &= F_0 \left(t, x, x', y, y'\right) + \mu F_1 \left(x, x', y, y'\right) x', \\
y'' &= G_{00} (x) + G_{01} (x) y + G_1 \left(x, u, y, y'\right) y',
\end{align*}
\]
where all functions in (3) are $C^\infty$, $\omega_1$, $\omega_2$, $\mu$ are positive constants and $\mu \gg 1$. This system generalizes those found in some mechanical models that take into account some kind of strong dissipative effect, such as viscous damping, van der Pol or Rayleigh dissipation. Let us write (3) as a singular first-order system. Thus
\[
\begin{align*}
x' &= u, \\
y' &= v, \\
v' &= -\omega_2^2 y + G_{00} (x) + G_{01} (x) y + G_1 \left(x, u, y, y'\right) u, \\
eu' &= \epsilon \left(-\omega_1^2 x + F_0 \left(t, x, u, y, v\right)\right) + F_1 \left(x, u, y, v\right) u,
\end{align*}
\]
where $\epsilon = 1/\mu$. This system can be studied using Invariant Manifolds, which main basic results are given in the Appendix and [5]. It is assumed that $F_1 \left(x, u, y, v\right) u = 0$ has an isolated solution given by $u = 0$. Using the notation of the Appendix, take $h_0 \left(t, x, y, v\right) = 0$. Let us suppose that
\[
F_1 \left(x, 0, y, v\right) \leq -2 \beta,
\]
where $\beta$ is an adequate positive constant. In view of Theorem A and (5), one concludes that (4) has a stable local integral manifold given by
\[
u = \epsilon h_1 \left(t, x, y, v\right) + O \left(\epsilon^2\right).
\]
Hence, by using (6) and (44), the singularly perturbed system (4) can be reduced to the following regularly perturbed system
\[
\begin{align*}
x' &= \epsilon h_1 \left(t, x, y, v\right) + O \left(\epsilon^2\right), \\
y' &= v, \\
v' &= -\omega_2^2 y + G_{00} (x) + G_{01} (x) y + eG_1 \left(x, u, y, v\right) h_1 \left(t, x, y, v\right) + O \left(\epsilon^2\right),
\end{align*}
\]
Now it is necessary to find $h_1 \left(t, x, y, v\right)$. In view of equation (43), one has that
\[
h_1 \left(t, x, y, v\right) = \frac{\left(\omega_1^2 x - F_0 \left(t, x, 0, y, v\right)\right)}{F_1 \left(x, 0, y, v\right)}
\]
So (7) can be rewritten as
\[
\begin{align*}
x' &= \epsilon \left(\omega_1^2 x - F_0 \left(t, x, 0, y, v\right)\right) + O \left(\epsilon^2\right), \\
y' &= v, \\
v' &= -\omega_2^2 y + G_{00} (x) + G_{01} (x) y + eG_1 \left(x, u, y, v\right) \left(\omega_1^2 x - F_0 \left(t, x, 0, y, v\right)\right) + O \left(\epsilon^2\right),
\end{align*}
\]
In the general setting of (4), some classes of general results can be obtained from (8). This will be shown in the next sections.

4. AN AUTONOMOUS CASE

Let us take $F_0 \left(t, x, x', y, y'\right) = H \left(x, x', y, y'\right)$, $G_{01} = 0$ and $G_1 = 0$. Then (8) can be rewritten as
\[
\begin{align*}
x' &= \epsilon \left(\omega_1^2 x - H \left(t, 0, 0, 0\right)\right) + O \left(\epsilon^2\right), \\
y' &= v, \\
v' &= -\omega_2^2 y + G_{00} (x) + O \left(\epsilon^2\right).
\end{align*}
\]
Let us assume $x \left(0\right) = A$, $y \left(0\right) = B$ and $v \left(0\right) = C$ as initial conditions of (9). In view of the regularity of the right hand side of the equations at (9) and from Theorem 3.3, pg.21 of [6] one has regular expansions of (9) in terms of the parameter $\epsilon$. Hence the following expansions hold
\[
\begin{align*}
x \left(t, A, B, C, \epsilon\right) &= x_0 \left(t, A, B, C\right) + x_1 \left(t, A, B, C\right) \epsilon + O \left(\epsilon^2\right), \\
y \left(t, A, B, C, \epsilon\right) &= y_0 \left(t, A, B, C\right) + y_1 \left(t, A, B, C\right) \epsilon + O \left(\epsilon^2\right), \\
v \left(t, A, B, C, \epsilon\right) &= v_0 \left(t, A, B, C\right) + v_1 \left(t, A, B, C\right) \epsilon + O \left(\epsilon^2\right).
\end{align*}
\]
Substituting (10) into (9), using the above initial conditions and solving two sets of linear differential equations one finds $x_0, y_0, v_0, x_1, y_1, v_1$. These computations are long but straightforward. Anyway, one gets
\[
\begin{align*}
x_0 \left(t, A, B, C\right) &= A, \\
y_0 \left(t, A, B, C\right) &= \frac{G_{00} (A)}{\omega_1^2} + C \sin \left(\omega_1 t\right) - \frac{C \sin \left(\omega_1 t\right)}{\omega_1^2} \cos \left(\omega_1 t\right), \\
v_0 \left(t, A, B, C\right) &= \frac{\left(\omega_1^2 A - \omega_1^2 B\right)}{C} \sin \left(\omega_1 t\right) + C \cos \left(\omega_1 t\right).
\end{align*}
\]
\[ x_1(t, A, B, C) = \int_0^t q(s, A, B, C) \, ds, \]
\[ y_1(t, A, B, C) = \frac{G_0'(A)}{\omega_2} \times \int_0^t \sin(\omega_2 s) \times \int_0^s q(z, A, B, C) \, dz \, ds, \]
\[ v_1(t, A, B, C) = G_0'(A) \times \int_0^t \cos(\omega_2 s) \times \int_0^s q(z, A, B, C) \, dz \, ds, \]
where
\[ q(t, A, B, C) = \left( \begin{array}{c} \omega_2^2 A \\ -H(p_0) \frac{\partial H}{\partial x}(p_0) \\ F_1 \left( A, 0, y_0(t, A, B, C), v_0(t, A, B, C) \right) \end{array} \right). \]
In this case, the Poincaré's Mapping is given by
\[ P(A, B, C, \epsilon) = \left( \begin{array}{c} x(\frac{2\pi}{\omega_2}, A, B, C, \epsilon) \\ y(\frac{2\pi}{\omega_2}, A, B, C, \epsilon) \\ v(\frac{2\pi}{\omega_2}, A, B, C, \epsilon) \end{array} \right). \]
It is well known if there is \((A_0, B_0, C_0)\) such that
\[ \frac{\partial P}{\partial \epsilon}(A_0, B_0, C_0, 0) = 0 \]
and
\[ \det(M(A_0, B_0, C_0)) \neq 0, \]
where
\[ M(A_0, B_0, C_0) = \frac{\partial^2 P}{\partial(A, B, C) \partial \epsilon}(A_0, B_0, C_0, 0), \]
then (9) has a \(\frac{2\pi}{\omega_2}\) periodic orbit given by
\[ \left( \begin{array}{c} x(t, A_0, B_0, C_0, \epsilon), \\ y(t, A_0, B_0, C_0, \epsilon), \\ v(t, A_0, B_0, C_0, \epsilon) \end{array} \right). \]
Moreover, if \(M(A_0, B_0, C_0)\) has two eigenvalues with norm lower than 0, then this orbit is asymptotically stable. If this matrix has at least one eigenvalue with norm greater than 0, this orbit is unstable.

Let us assume that there is \(A_0\) such that
\[ F_1 \left( A_0, 0, \frac{G_0'(A_0)}{\omega_2^2}, 0 \right) < 0 \]
and
\[ H \left( A_0, 0, \frac{G_0'(A_0)}{\omega_2^2}, 0 \right) = \omega_2^2 A_0. \]

Take
\[ (A_0, B_0, C_0) = \left( A_0, \frac{G_0'(A_0)}{\omega_2^2}, 0 \right). \]
After a long computation of \(M(A_0, B_0, C_0)\), which will be denoted by \(J\), and using (16), (17) one obtains
\[ J_{11} = -\frac{2\pi}{k_2\omega_2} \left( \frac{2\omega_2^2 \frac{\partial^2 H}{\partial x^2}(p_0) + 3G_0'(A_0) \frac{\partial H}{\partial y}(p_0)}{\omega_2^1 \omega_2^2} \right), \]
\[ J_{12} = J_{13} = 0, \]
\[ J_{21} = \frac{\pi G_0'(A_0)}{k_2 \omega_2^3} \left( \frac{2\omega_2^2 \frac{\partial^2 H}{\partial x^2}(p_0) + 3G_0'(A_0) \frac{\partial H}{\partial y}(p_0)}{\omega_2^1 \omega_2^2} \right), \]
\[ J_{22} = -\frac{\pi G_0'(A_0)}{k_2 \omega_2^3} \frac{\partial H}{\partial y}(p_0), \]
\[ J_{23} = -\frac{\pi G_0'(A_0)}{k_2 \omega_2^3} \frac{\partial H}{\partial v}(p_0), \]
\[ J_{31} = -G_0'(A_0)J_{23}, J_{32} = \omega_2^2J_{23}, J_{33} = -J_{22}, \]
where \(k = F_1(p_0)\) and
\[ p_0 = \left( A_0, 0, \frac{G_0'(A_0)}{\omega_2^2}, 0 \right). \]
Using (19) one gets that the eigenvalues \(\lambda_1, \lambda_2, \lambda_3\) of \(J\) are given by
\[ \lambda_1 = -\lambda_2 = \frac{\pi G_0'(A_0) \sqrt{\Delta(A_0)}}{k_2 \omega_2^3}, \]
and
\[ \lambda_3 = \frac{2\pi}{k_2 \omega_2^3} \left( \frac{2\omega_2^2 \frac{\partial^2 H}{\partial x^2}(p_0) + 3G_0'(A_0) \frac{\partial H}{\partial y}(p_0) - \Delta(A_0)}{\omega_2^1 \omega_2^2} \right), \]
where
\[ \Delta(A_0) = \frac{\partial H}{\partial y}(p_0)^2 + \omega_2^2 \frac{\partial \Delta}{\partial v}(p_0)^2. \]
Assume that
\[ \lambda_1, \lambda_3 \neq 0. \]
It follows from this condition and (17) that (14) and (15) hold. Hence there is a \(\frac{2\pi}{\omega_2}\) periodic orbit of (9). Moreover, since (16) is valid and the foregoing orbit is near \(\left( A_0, \frac{G_0'(A_0)}{\omega_2^2}, 0 \right)\) one concludes that (5) holds at each point of this orbit for \(\beta\) adequate chosen. Since \(\lambda_1 \neq 0\) then from
(20) it follows that one of the eigenvalues of $J$ is greater than zero then the above orbit is unstable. Even with all above hypotheses, this result is a very general one. Summing up, take

$$
\begin{align*}
    x'' + \omega^2_1 x &= H(x, x', y, y') + \mu F_1(x, x', y, y'), \\
    y'' + \omega^2_2 y &= G_0(x).
\end{align*}
$$

(23)

If there is $A_0$ such that (16), (17) and (22) hold then (23) has a unstable hyperbolic periodic orbit.

One can see that if $H(0, 0, y, v) = 0$ then $(0, y_0(t))$, where $y_0' + \omega^2_0 y_0 = G_0(0)$, is a semi-trivial solution of (23).

Let us take (1) with $\delta = \epsilon = \kappa = 0$ and $\delta_0 = \mu$. Then we have

$$
\begin{align*}
    x'' + \omega_1^2 x &= -\gamma x^3 - ax y - \mu x', \\
    y'' + \omega_2^2 y &= -b x^2.
\end{align*}
$$

(24)

Let us assume $a b - \gamma \omega^2 > 0$. One obtains $A_0 = 0$, $\pm \sqrt{a b - \gamma \omega^2}$. And from above results one concludes that (24) has two unstable hyperbolic periodic orbits. The case $A_0 = 0$ will be investigated in the next subsection. If $a b - \gamma \omega^2 \leq 0$ one has $A_0 = 0$ too.

For (24) the reduced system is given by

$$
\begin{align*}
    x' &= \epsilon (-a x y - \gamma x^3 - \omega_1^2 x) + O(\epsilon^2), \\
    y' &= v, \\
    v' &= -\omega_2^2 y - b x^2.
\end{align*}
$$

(25)

5. QUENCHING FOR (24)

In this section we discard the term $O(\epsilon^2)$ of (25) and the dynamics of the following system is investigated.

$$
\begin{align*}
    y' &= v, \\
    v' &= -\omega_2^2 y - b x^2, \\
    x' &= \epsilon (-a x y - \gamma x^3 - \omega_1^2 x),
\end{align*}
$$

(26)

It follows that $(0, 0, 0)$ is an equilibrium point of (26). The linearization of (26) at this point gives

$$
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
-\omega_2^2 & 0 & -\epsilon \omega_1^2
\end{pmatrix}.
$$

(27)

From (27) and Theorem 1, pg.4 of [7] the system (26) has a center manifold $x = Q(y, v)$. Moreover, since $x = 0$ is an invariant manifold of (26) one can take $Q = 0$. From this and Remark 2.16, pg. 322 of [8] it follows that (26) and the system

$$
\begin{align*}
    z' &= w, \\
    w' &= -\omega_2^2 z, \\
    x' &= -\epsilon \omega_1^2 x,
\end{align*}
$$

(28)

are topologically equivalent. Hence all periodic solutions of (26) are stable.

Here there is an interesting question. It is clear that $(0, 0)$ is a stable equilibrium point of (28)$_{1,2}$. Hence from Theorem 2b, pg.4 of [7] for each solution $(y, v, x)$ of (26) there is a solution $(\rho_0 \cos(\omega_2 t + \varphi_0), -\omega_2 \rho_0 \sin(\omega_2 t + \varphi_0))$ of (28)$_{1,2}$ such that as $t \to \infty$

$$
\begin{align*}
    y &= \rho_0 \cos(\omega_2 t + \varphi_0) + O(e^{-\nu t}), \\
    v &= -\omega_2 \rho_0 \sin(\omega_2 t + \varphi_0) + O(e^{-\nu t}), \\
    x &= O(e^{-\nu t}),
\end{align*}
$$

(29)

where $\nu$ is a positive constant. A natural question is to ask for $\rho_0$. Here we are going to sketch an informal approach for that. Let us do the following change of variables $(y, v, x) \to (\delta y, \delta v, \delta x)$ at (26), thus

$$
\begin{align*}
    y' &= v, \\
    v' &= -\omega_2^2 y - b \delta x^2, \\
    x' &= \epsilon (-a \delta x y - \gamma \delta^2 x^3 - \omega_1^2 x).
\end{align*}
$$

(30)

Now take

$$
\begin{align*}
    y &= r \cos(\theta), \\
    v &= -\omega_2 \sin(\theta).
\end{align*}
$$

(31)

Using (31) in (30) the following system is obtained

$$
\begin{align*}
    r' &= \delta \sin(\theta) b x^2, \\
    \theta' &= \omega_2 + \delta \cos(\theta) b x^2, \\
    x' &= (-\epsilon \omega_2^2 b \delta \cos(\theta) a r ) x - \delta^2 \epsilon \gamma x^3.
\end{align*}
$$

(32)

Then

$$
\begin{align*}
    \frac{dx}{d\theta} &= \frac{(-\epsilon \omega_2^2 - \delta \cos(\theta) a r) x - \delta^2 \epsilon \gamma x^3}{\omega_2 + \delta \cos(\theta) b x^2}, \\
    \frac{d\theta}{d\theta} &= \frac{-\delta \sin(\theta) b x^2}{\omega_2 + \delta \cos(\theta) b x^2}.
\end{align*}
$$

(33)

Now, we can use Regular Perturbation Theory at (33). Take

$$
\begin{align*}
    x &= x_0 + x_1 \delta + O(\delta^2), \\
    r &= r_0 + r_1 \delta + O(\delta^2).
\end{align*}
$$

(34)

Substituting this at (33) one gets the following systems

$$
\begin{align*}
    \frac{dx_0}{d\theta} &= -\epsilon \omega_2^2 x_0 \omega_2, \\
    \frac{dr_0}{d\theta} &= 0,
\end{align*}
$$

(35)

and

$$
\begin{align*}
    \frac{dx_1}{d\theta} &= -\epsilon \omega_2^2 x_1 - \epsilon \cos(\theta) a x_0 x_0 + \frac{\epsilon \cos(\theta) b x_0^2 x_0}{r_0 \omega_2^3}, \\
    \frac{dx_2}{d\theta} &= \sin(\theta) b x^2 \omega_2^3.
\end{align*}
$$

(36)

Taking the initial conditions $x(0) = A$, $r(0) = R$ and using (34) one has $x_0 = A$, $x_1 = 0$, $r_0 = R$ and $r_1 = 0$. From these initial conditions, (35), (36) and (34) one gets

$$
\begin{align*}
    x(\theta) &= e^{-\epsilon \omega_2^2 \theta} A + \left( x_{11}(\theta) e^{-\epsilon \omega_2^2 \theta} + x_{12}(\theta) e^{-3 \epsilon \omega_2^2 \theta} \right) \delta + O(\delta^2),
\end{align*}
$$

(37)
and

\[
    r(\theta) = R + \left( \frac{b A^2}{\omega_2^2 + 4 \epsilon^2 \omega_1^2} + r_{11}(\theta) e^{\frac{2 \epsilon \omega_1^2}{-2}} \right) \delta + O(\delta^2),
\]

(38)

where \(x_{11}, x_{12}, r_{11}\) are 2\(\pi\) periodic functions. Since \(r(t) = \sqrt{y(t)^2 + (v(t)/\omega_2)^2}\) take \(R = \sqrt{y(0)^2 + (v(0)/\omega_2)^2}\). It follows from (29) that \(\lim_{t \to \infty} r(t) = \rho_0\). Analogous behavior must be hold for the reduced system (33). Hence, in view of (38) it is expected that

\[
    \rho_0 = R + \left( \frac{b A^2}{\omega_2^2 + 4 \epsilon^2 \omega_1^2} \right) \delta + O(\delta^2).
\]

(39)

An immediate conclusion from (39) is that if \(b > 0\) then the final amplitude will be greater than the initial one and if \(b < 0\) will be lesser. Let us check these qualitative conclusions numerically.

In (26) take \(\epsilon = 0.01, \gamma = 0.5, a = b = -1, \omega_1 = 1, \omega_2 = 2\) and the initial conditions \(y(0) = 1, v(0) = 0, x(0) = 0.2\). Thus we have

\[
    \text{Figure 2 – Stable Periodic Orbit: Case } A < 1
\]

From Figure 2 one has the amplitude, which is equal to \(\sqrt{y(0)^2 + (v(0)/\omega_2)^2}\), of the orbit in blue is almost the same of the periodic one in red. This agrees with (39) when \(A = x(0)\) is small.

Let us take \(\epsilon = 0.1, \gamma = 1, a = b = -1, \omega_1 = 1, \omega_2 = 2\) and the initial conditions \(y(0) = 1, v(0) = 0, x(0) = 1\). Hence the following graphic at Figure 3 is obtained.

\[
    \text{Figure 3 – Stable Periodic Orbit: Case } b < 0
\]

The final amplitude is lesser than the initial one and this is in agreement with (39). Similar result holds for the case \(b > 0\) as it is shown in the Figure 4. In this case all parameters and initial conditions are the same of those used for Figure 3 except for \(b\) which is equal to 1.

\[
    \text{Figure 4 – Stable Periodic Orbit: Case } b > 0
\]

Taking the above parameters directly at (24) and making projections onto \(yvx\)-space one gets a graphic similar to Figure 3. For \(yvu\)-space we have Figure 5. It is clear that these computations are in agreement with the foregoing results. It is worth to note that reductions on the integral manifolds were not used in these projections.
6. CONCLUSIONS

We have investigated the dynamics of some singularly perturbed problems that come from Mechanics. Some general conclusions were obtained for the system (23) A quenching has been found out at the system (24). Of course it will be necessary to take into account the non-autonomous case as well as the case \( G_1 \neq 0 \). This will be done in other papers.

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APPENDIX: INTEGRAL MANIFOLDS

In this section it will be given basic results about Integral Manifolds, which have been used in this paper. Our main reference is [5] and references therein.

Firstly, we will give the definition of Integral Manifold. Take the differential equation:

\[
X' = N(t, X),
\]

where \( X, N \in \mathbb{R}^n \). The set \( \mathcal{M} \subset \mathbb{R} \times \mathbb{R}^n \) is said to be an integral manifold if for \( (t, X(t)) \), \( X(t_0) = X_0 \), is in \( \mathcal{M} \) for all \( t \in \mathbb{R} \) and \( X_0 \in \mathcal{M} \). If \( (t, X(t)) \in \mathcal{M} \) for only a finite interval of time, then \( \mathcal{M} \) is said to be a local integral manifold.

In the next theorems we are interested in to study singularly perturbed systems and invariant manifolds will be given as graph of adequate mappings. Indeed, consider the following system of differential equations:

\[
\begin{align*}
\dot{x} &= f(t, x, y, \epsilon), \\
\epsilon \dot{y} &= g(t, x, y, \epsilon),
\end{align*}
\]

where \( x, f \in \mathbb{R}^n, y, g \in \mathbb{R}^m, t \in \mathbb{R}, \) and \( \epsilon \) is a small parameter.

In the next theorem is proved the existence and some properties of the invariant manifolds.

Theorem A (Existence and Properties of Integral Manifolds)

Suppose the following hypotheses hold:

- \( A_1 \): The equation \( g(t, x, y, 0) = 0 \) has an isolated solution \( y = h_0(t, x), \forall t \in \mathbb{R}, \forall x \in \mathbb{B}_x \), where \( \mathbb{B}_x \) is an open neighborhood of \( x \) in \( \mathbb{R}^n \).
- \( A_2 \): The functions \( f, g, \) and \( h_0 \) are \( C^2 \) at \( (t, x, \epsilon) \in \mathbb{R} \times \mathbb{B}_x \times \{0, \epsilon_0\} \), and for \( ||y - h_0(t, x)|| \leq \mathcal{P}_y \), where \( \epsilon_0 \) and \( \mathcal{P}_y \) are positive real constants.
- \( A_3 \): The eigenvalues \( \lambda_i = \lambda_i(t, x), i = 1, 2, \ldots, m \) of the matrix \( Z(t, x) = (\partial g/\partial y)(t, x, h_0(t, x), 0) \) satisfy the inequality

\[
\Re \lambda_i \leq -2\beta < 0 \quad \forall (t, x) \in \mathbb{R} \times \mathbb{B}_x. \tag{41}
\]

Then, there exists \( \epsilon_1 \leq \epsilon_0 \) such for all \( \epsilon \in [0, \epsilon_1] \), the singularly perturbed system (40) has a \( m \)-dimensional local integral manifold

\[
\mathcal{M}_\epsilon : y = h_0(t, x) + H(t, x, \epsilon) = h(t, x, \epsilon), \tag{42}
\]

where \( h(t, x, \epsilon) \) is defined for all \( x \in \mathbb{B}_x \) and \( \epsilon \leq \epsilon_1 \), is a \( C^1 \) mapping and \( (t, x, \epsilon) \) satisfies the inequalities

\[
||H(t, x, \epsilon)|| \leq \rho_1(\epsilon)
\]

\[
||H(t, x_1, \epsilon) - H(t, x_2, \epsilon)|| \leq \rho_2(\epsilon)||x_1 - x_2||,
\]

where \( \rho_1(\epsilon) \to 0 \) and \( \rho_2(\epsilon) \to 0 \) as \( \epsilon \to 0 \). The function \( h(t, x, \epsilon) \) is a \( C^1 \) mapping and satisfies the so-called manifold equation

\[
\epsilon \frac{\partial h}{\partial t} + \epsilon \frac{\partial h}{\partial x} f(t, x, h, \epsilon) = g(t, x, h, \epsilon),
\]

which is obtained by substituting \( y \) by \( h \) in (40). On this manifold, the flow of (40) is governed by the \( n \)-dimensional reduced system

\[
x' = f(t, x, h(t, x), \epsilon). \tag{44}
\]

Moreover, if for \( x \in \mathbb{B}_x \) and \( p \) integer one has \( f(t, x, y, \epsilon) \in C^{p+1}, \) \( f(t, x, y, \epsilon) \in C^{p+2}, \) and \( h_0(t, x) \in C^{p+1}, \) then \( h \in C^p \).

From the differentiability of \( h \) it is obtained that

\[
\frac{h(t, x, \epsilon) - h_0(t, x)}{\epsilon} = h_1(t, x, \epsilon) + O(\epsilon^2).
\]

Results on stable and unstable orbits can be obtained from the following theorem:

Theorem B (Stability)

Under the same assumptions of Theorem A, there exists \( \epsilon^* \leq \epsilon_1 \) such that \( \forall \epsilon \in [0, \epsilon^*] \), and for any solution \( x(t), y(t), x(t_0) = x_0, y(t_0) = y_0 \) of (40) with sufficiently small \( ||y_0 - h(t_0, x_0, \epsilon)|| \), there is a solution \( m(t) \), \( m(t_0) = m_0 \) of (44) satisfying

\[
x(t) = m(t) + \phi_1(t),
\]

\[
y(t) = h(t, m(t), \epsilon) + \phi_2(t),
\]

where \( \phi_i(t) = O(e^{-\beta(t-t_0)}) \) as \( \phi_2(t) \to \infty, \) \( i = 1, 2 \).

Of course Theorem B can be used to get an instability result if \( m(t) \) is an unstable periodic orbit.
REFERENCES


