DESCRIPTING A PHASE TRANSITION IN THE DYNAMICS OF A PARTICLE MOVING IN A TIME-DEPENDENT POTENTIAL WELL

Diogo Ricardo Costa¹, Mário Roberto Silva², Juliano A. de Oliveira³, Edson D. Leonel⁴

¹UNESP, Rio Claro, Brazil, diogo_cost@hotmail.com
²UNESP, Rio Claro, Brazil, marsilva@rc.unesp.br
³UNESP, Rio Claro, Brazil, julianoantonio@gmail.com
⁴UNESP, Rio Claro, Brazil, edleonel@rc.unesp.br

Abstract: Some dynamical properties for a classical particle confined in an infinitely deep box of potential containing a periodically oscillating square well are studied. The dynamics of the system is described by a two dimensional non-linear area preserving mapping for the variables energy and time. The phase space is mixed and the chaotic sea is described using scaling arguments. Thus, critical exponents are obtained near a transition from integrability to non-integrability. The formalism is robust and can be extent to many different kinds of mappings.

Keywords: chaos, well, potential.

1. INTRODUCTION

Dynamical systems described by mappings have been considered widely along the last years [1, 2]. In special and for the most simple cases, i.e. for systems with 1 and 1/2 degrees of freedom, that correspond to a time perturbation in a system with one-degree of freedom, the description of Hamiltonian systems lead many times to two dimensional non-linear area preserving mappings. Many different applications of the formalism of two-dimensional area preserving mappings are observed, particularly in the study of magnetic field lines in toroidal plasma devices with reversed shear (like tokamaks) [3–6], waveguide [7–11], Fermi acceleration [12], billiards [13–15] and many others generalizations [16–19].

In this paper we consider the dynamics of a system which is a 1 and 1/2 degrees freedom problem. The model consists of a classical particle which is confined inside an infinitely deep potential well which contains a time periodically moving square well. The Hamiltonian that describes the model is \( H(x, p, t) = p^2/(2m) + V(x, t) \) where \( V(x, t) = V_0(x) + V_1(x, t) \), and \( x, p \) and \( t \) correspond to position and momentum coordinates and time respectively. The potential \( V_0(x) \) denotes the integrable part of the Hamiltonian while \( V_1(x, t) \) leads to the non-integrable part. As we will see in the next section, the potential \( V_1(x, t) \) is controlled by three relevant control parameters. If they are changed accordingly, a phase transition from integrability to non-integrability can be observed. Among other things we discuss along the paper, this transition is basically the main focus of the present work.

The dynamics of the model is described by a two-dimensional non-linear area preserving mapping for the variables energy and time. The phase space of the model is of mixed type in the sense that Kolmogorov-Arnold-Moser (KAM) islands are observed surrounded by a chaotic sea which is characterized by a positive Lyapunov exponent. The size of the chaotic sea depends on the control parameters and is limited by a set of invariant tori (also called as invariant spanning curves) which prevents the energy of the particle to grow unlimited. Thus, if the law which controls the time perturbation of the moving well is smooth enough, Fermi acceleration [20] (unlimited energy growth of the particle) is not observed (see Ref. [21] for similar discussion in one-dimensional Fermi accelerator models and Ref. [22] where it happens for specific range of control parameters). The size of the chaotic sea is strongly dependent on the control parameters of the model. Thus if the amplitude of oscillation of the moving well is fixed as zero, the system is integrable and phase space exhibits only periodic fixed points or straight lines (see Ref. [23] for more discussion). However, if it is slightly different from zero, the phase space birth as mixed and the position of the lowest energy invariant spanning curve, which limit the chaotic sea, depends on the control parameters. Average properties of the chaotic sea however can be described by using the so called scaling approach. Thus, near the phase transition from integrability to non-integrability, critical exponents characterize the law which describe how average properties of the chaotic sea, like average energy of the particle or the deviation around the average energy, behave near such a criticality. Moreover, classes of universality [24] can be classified according to the
This paper is organized as follows. In Sec. 2 we describe the model under consideration and four different potential shapes which lead to the same characterization of the problem. Our numerical results and the critical exponents are described in Sec. 3. Final remarks and conclusions are drawn in Sec. 4.

2. THE MODEL AND THE MAP

In this section we discuss the model and the corresponding mapping which describes the dynamics of a classical particle. The model consists of classical particle which is confined inside a box of an infinitely deep potential which contains an oscillating square well in the middle. In this paper, we assume the oscillations of the bottom move periodically in time according to a cosine function. A typical sketch of the potential is shown in Fig. 1. Along the region $b$ (where the potential is constant and equal to $V_0$) and the region $a$ (where the potential is not constant), the velocities of the particle are constants, because there are not forces acting in the particle. The total energy, is obtained by summation of kinetic and potential energy. When the particle changes from one region to another, it experiences and abrupt change in its kinetic energy.

We stress however that other different kinds of potential shapes also lead to the same description of the dynamics. One can consider a chain of infinitely many and symmetric oscillating square wells with their bottoms moving periodically and synchronized in time. (a) A single oscillating square well. (b) A chain of infinitely many and symmetric oscillating square wells with their bottoms moving periodically and synchronized in time. (c) A step potential whose bottom moves periodically in time.

In this paper however, we consider the description of the model as shown in Fig. 1. The potential $V(x, t)$ is given by

$$V(x, t) = \begin{cases} \infty, & \text{if } x \leq 0 \text{ or } x \geq (a+b) \\ V_0, & \text{if } 0 < x < \frac{a}{2} \text{ or } (a+\frac{b}{2}) < x < (a+b) \\ V_1 \cos(\omega t), & \text{if } \frac{a}{2} \leq x \leq (a+\frac{b}{2}) \end{cases}$$

where the control parameters $a, b, V_0, V_1$ and $\omega$ are constants. Considering the symmetry of the problem the two-dimensional mapping is obtained upon entrance of the particle in the oscillating square well. To construct the map, we suppose that at the time $t = t_n$, the particle has energy $E = E_n$ and it is ready to enter the oscillating square well. When the particle enters the well it suffers an abrupt change in its kinetic energy so that we have $E_n' = E_n - V_1 \cos(\omega t_n) = \frac{1}{2}mv_n'^2$ where $|v_n'| = \sqrt{2K_n/m}$ is constant. Arriving in the other side of the well, the energy of the particle is

$E_n'' = E_n' + V_1 \cos(\omega(t_n + \Delta t_n'))$ where $\Delta t_n' = a/|v_n'|$. Thus, if the particle does not have enough energy to escape the well and it is reflected backwards i.e. $E_n'' \leq V_0$. It may thus suffers many other successive reflections until escapes the well and $E_n''$ may be redefined more generally as $E_n'' = K_n'' + V_1 \cos(\omega(t_n + i\Delta t_n''))$ where $i$ is the smallest positive integer number which matches the condition $E_n'' > V_0$, condition which assures the particle to escape. Once the particle escapes the oscillating well, it travels towards the infinity potential boundary of the box, suffers an elastic reflection from it and moves backwards towards a next entrance to the oscillating square well, thus the new energy of the particle is given by $E_{n+1} = E_n + V_1 \{\cos(\omega(t_n + i\Delta t_n'')) - \cos(\omega t_n)\}$. The time at the next entrance is written as $t_{n+1} = t_n + i\Delta t_n'' + \Delta t_n'''$ where $\Delta t_n''' = b/|v_n''|$ with $|v_n''| = \sqrt{2K_n''/m}$ and $K_n'' = E_{n+1} - V_0$. 

Figure 1 – Sketch of the potential considered.

Figure 2 – (a) A chain of infinitely many and symmetric oscillating square wells with their bottoms moving periodically and synchronized in time. (b) A single oscillating square well. (c) A step potential whose bottom moves periodically in time.
We see that there are too many control parameters which are not all relevant to describe the dynamics, five in total, namely $a$, $b$, $V_0$, $V_1$ and $\omega$. Defining dimensionless variables we obtain $\delta = V_1/V_0$, $r = b/a$, $e_n = E_n/V_0$, $N_c = \omega/(2\pi)(a/\sqrt{2V_0/m})$ and measure the time in terms of the number of oscillations of the moving well, $\phi = \omega t$. The parameter $N_c$ corresponds to the number of oscillations that the square well completes in a time $t = a/\sqrt{2V_0/m}$.

With this set of new control parameters, the mapping is written as

$$ T : \begin{cases} e_{n+1} = e_n + \delta[\cos(\phi_n + i\Delta\phi_n) - \cos\phi_n] \\ \phi_{n+1} = \phi_n + i\Delta\phi_n + \Delta\phi_b \mod 2\pi \end{cases}, $$

(2)

where the auxiliary variables are given by

$$ \Delta\phi_n = \frac{2\pi N_c}{\sqrt{e_n - \delta \cos(\phi_n)}}, \quad \Delta\phi_b = \frac{2\pi N_c r}{\sqrt{e_{n+1} - 1}}. $$

When the particle stays confined in the oscillating square well with energy at its boundary $e \leq 1$, such a trapping would be longer or not and the distribution of successive reflection numbers obeys a power law, as shown in Fig. 3. A power law fitting furnish that the exponent is around $-3$, as that one obtained for a time dependent potential barrier [26, 27].

Since the determinant of the Jacobian matrix is equal to the unity, the mapping (2) is area preserving. The phase space of the model is mixed containing both KAM islands, chaotic sea and invariant spanning curves, as one can see in Fig. 4.

Given the expression of the mapping, we can also obtain the corresponding fixed points. The period one fixed points, without multiple reflections, are obtained by $e^* = e_n = \epsilon^*$ and $\phi^* = \phi_n + 2m\pi = \phi^*$, where $m$ is a non negative integer number. Considering the periodic functions of the mapping, it is necessary to apply two different procedures to obtain the fixed points $[e^*, \phi^*]$: (i) obtained analytically by the expressions

$$ \left[\left(\frac{N_c r}{m - k}\right)^2 + 1, \arccos\left(\frac{1}{\delta\left(e - \left(\frac{N_c}{k}\right)^2\right)}\right)\right], $$

(3)

and

$$ \left[\left(\frac{N_c r}{m - k}\right)^2 + 1, 2\pi - \arccos\left(\frac{1}{\delta\left(e - \left(\frac{N_c}{k}\right)^2\right)}\right)\right]; $$

(4)
onde $k$ arises when we make $e_{n+1} = e_n$ and this results that the arguments of the cosine functions must differ from a integer multiple of $2\pi$, i.e. $\Delta\phi_n = k2\pi$. Another possibility, is make $\Delta\phi_n = k2\pi - 2\phi$ originating the procedure (ii) that gives the fixed points after a numerical solution of

$$1 + \left(\frac{\pi N_c r}{\pi(m-k)+\phi}\right)^2 - \delta \cos(\phi) - \left(\frac{\pi N_c}{\pi k-\phi}\right)^2 = 0, \quad (5)$$

where $m > k$, $m$ and $k$ are constants larger than 1. Solution of Eq. (5) furnishes $\phi$ numerically. The corresponding energy is given by

$$e = \left(\frac{\pi N_c r}{\pi(m-k)+\phi}\right)^2 + 1. \quad (6)$$

The characterization of the fixed points is made by the eigenvalues of the Jacobian matrix $J$ [23]. Thus if the evaluation of the eigenvalues at the fixed point produces $(\text{Tr} J)^2 > 2$, the fixed point is said to be hyperbolic. On the other hand, for the case of $(\text{Tr} J)^2 < 2$, the fixed points are classified as elliptic. The fixed points for mapping (2) are identified in Fig. 4 by circles (elliptic fixed point) and crosses (hyperbolic fixed point). The red (gray) circles (crosses) correspond to elliptic (hyperbolic) fixed points obtained by Eqs. (3) and (4) while the blue (black) are obtained via Eqs. (5) and (6).

3. NUMERICAL RESULTS

Let us now discuss our numerical results. We start the discussion presenting the Lyapunov exponents characterizing the chaotic properties of the chaotic sea at low energy. It is well known that the Lyapunov exponents are widely used for the characterization of chaotic properties in dynamical systems. Basically the procedure to obtain the Lyapunov exponents consists in verify if two nearby initially trajectories diverge exponentially for an infinitely long time. If the system exhibits at least one positive Lyapunov exponent, then it has chaotic components. The Lyapunov exponents can be obtained by [28]

$$\lambda_j = \lim_{n \to \infty} \frac{1}{n} \ln |\Lambda_j^{(n)}|, \quad j = 1, 2 \quad (7)$$

where $\Lambda_j^{(n)}$ are the eigenvalues of the matrix $M = \prod_{j=0}^{n-1} J_0(e_i, \phi_i)$ and $J_i$ is the Jacobian matrix of our system. It is shown in Fig. 5(a) a plot of the positive Lyapunov exponent as function of $n$ for six different initial conditions randomly chosen along the chaotic sea. The control parameters used were $r = 1$, $N_c = 500$ and $\delta = 0.5$. After an initial fluctuation, the positive Lyapunov exponent converges to a constant value for large enough $n$. It is also important to obtain the behavior of $\lambda$ as function of the control parameters $N_c$, $r$ and $\delta$. Figure 5(b) shows a plot of $\lambda \times N_c$ for fixed $r = 1$ and $\delta = 0.5$. One can see that the positive Lyapunov exponent varies from $\lambda \approx 0.5$ for $N_c = 1$ up to $\lambda \approx 2$ for $N_c = 10^3$. It also has a monotonic tendency to growth as function of $N_c$. Note however that increasing $N_c$ corresponds to raising the number of oscillations of the well and consequently increasing the randomness of the system, therefore leading to an increase in the Lyapunov exponent. A plot of $\lambda \times \delta$ is shown in Fig. 5(c). The control parameters used were $r = 1$ and $N_c = 33.18$. One can also see that small values of $\delta$, which correspond to very small fluctuations of the oscillating square well produce a large Lyapunov exponent. A minimum value of $\lambda \approx 1.4$ was observed for $\delta \approx 0.2$. Finally, a plot of $\lambda \times r$ is shown in Fig. 5(d) for fixed $\delta = 0.5$ and $N_c = 33.18$. Since the control parameter $r = b/a$, enlarging $r$ for a fixed $N_c$ corresponds to enlarging $b$ thus increasing the distance from the well up to the box of potential. Such an increase leads to a long flight of the particle until next entrance in the oscillating square well therefore yielding in an increase of the number of oscillations of the moving well and consequently increasing the randomness of the system. The sudden jumps in the behavior of the Lyapunov exponent are explained as the destruction of invariant spanning curves leading to a joint of different chaotic regions (see for example Ref. [22] for a discussion in Fermi-Ulam model and Ref. [29] for time dependent square well).

Let us now address properly the scaling description of the chaotic sea. The position of the lowest energy invariant spanning curve (invariant tori) depends on the control parameters. As they change, such a position also varies, given rise to the chaotic sea to enlarge or reduce. Our main interest is to characterize the behavior of the average energy and hence the deviation of the average energy of the chaotic sea as function of the control parameters. To do so, we define the average

![Figure 5](image-url)
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Figure 6 – (Color online) Plot of $\omega \times n$ for fixed $r = 1$, $\delta = 0.5$ and three different $N_c$, namely $N_c = 100$, $N_c = 500$ and $N_c = 1000$.

energy as

$$\tau(n, \delta, N_c, r) = \frac{1}{N_c} \sum_{i=1}^{N_c} \epsilon_i,$$

and hence the deviation of the average energy is given by

$$\omega(n, \delta, N_c, r) = \frac{1}{M} \sum_{j=1}^{M} \sqrt{\epsilon_j^2(n, \delta, N_c, r) - \bar{\epsilon}^2(n, \delta, N_c, r)},$$

where $M$ denotes an ensemble of different initial conditions. Figure 6 shows a plot of three different curves of $\omega$ as function of $n$ obtained for different values of the control parameter $N_c$. One can see that, after an initial and short transient, $\omega$ starts growing according to a power law and eventually after reaching a characteristic crossover time, say $n_x$, it bends towards a regime of saturation for long enough $n$. Similar behavior is observed when the control parameters $\delta$ and $r$ are varied. Based on such kind of behavior, we propose the following scaling hypotheses:

- (i) For $n \ll n_x$, the behavior of $\omega$ can be described as

$$\omega(n, \delta, N_c, r, \delta) \propto [n \delta^2]^{\beta},$$

where $\beta$ is a critical exponent;

- (ii) For $n \gg n_x$, $\omega_{sat}$ is given by

$$\omega_{sat}(n \delta^2, N_c, r, \delta) \propto N_c^{-\alpha_1} r^{\alpha_2} \delta^{\alpha_3},$$

with both $\alpha_i$ for $i = 1, 2, 3$ are critical exponents;

- (iii) The characteristic crossover $n_x$ is written as

$$n_x(n \delta^2, N_c, r, \delta) \propto N_c^{-z_1} r^{z_2} \delta^{z_3},$$

and $z_i$ with $i = 1, 2, 3$ denote the dynamic exponent.

The critical exponents can be obtained if the behavior of $\omega_{sat}$ and $n_x$ are obtained as function of the control parameters. We have made extensive numerical investigations and found that $\beta \approx 0.5$. The critical exponents obtained as function of $N_c$ are shown in Fig. 7.

Fitting the curves shown in Fig. 7(a,b) by power law, we obtain that $\alpha_1 = 0.669(1)$ and $z_1 = 1.31(1)$. Extending the procedure to to the variable $r$, as shown in Fig. 7(c,d) we obtain $\alpha_2 = 0.295(4)$ and $z_2 = 0.59(1)$. Finally, the critical exponents as function of the control parameter $\delta$ are shown in Fig. 7(e,f) respectively. We see that there exist two different sets of values for $\alpha_3$ and $z_3$. The explanation for such a difference is because there is a sudden destruction of an invariant spanning curve that separate two different chaotic regions (see for example Ref. [22] for occurrence of such a destruction on the Fermi-Ulam model). After the destruction and when the two chaotic components join each other, the values of the critical exponents change substantially. Thus, before the destruction, say $\delta < 0.2$, the two sets of critical exponents obtained were $\alpha_3 = 0.721(7)$ and $z_3 = -0.58(3)$.
elastic and no damping forces are present) is described via a generic two-dimensional area preserving mapping \([31]\) and periodically corrugated waveguide \([7, 8]\); (ii) cases, we consider the following examples: (i) Fermi-Ulam model \([30]\) and periodically oscillating square well. The two dimensional area preserving mapping that describes the dynamics of the model has three relevant control parameters. Average properties of the chaotic sea at low energy level were investigated under the randomness of the model, a similar behavior happens as when the Fermi-Ulam model. The models in this model. The critical exponents obtained were the same, within an uncertainty error. Thus the two different models exhibit the same critical exponents and therefore belong to the same class of universality for such a phase transition.

4. SUMMARY AND CONCLUSIONS

We have studied the dynamics of a classical particle confined in a box of infinitely deep potential containing a periodically oscillating square well. The two dimensional area preserving mapping that describes the dynamics of the model has three relevant control parameters. Average properties of the chaotic sea at low energy level were investigated under scaling approach as function of the three control parameters. We found critical exponents and validate our scaling hypotheses by a merger, after a suitable rescaling of the axis, of different curves of \(\omega\) onto a single and universal plot. Such kind of behavior is typical of systems experiencing phase transition. We found also the Lyapunov exponents...
and showed that the system studied has chaotic components. Fixed point were also obtained and characterized.

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