Klein – Gordon – Dirac Operators and Transcendental Quaternions

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Abstract

In this work we build up an analog of the classical De Moivre relation for general quaternions and octonions numbers, and present connections between quaternion and octonion transcendent functions to differential operators of the Fueter theory.

Key Words: Fueter theory, Hipercomplex functions, De Moivre relation.

Introduction

Since 1843, when Hamilton discovered the quaternions, have been dealing with this extraordinary numbers which provide useful insights on the structure of different areas of the mathematics or on the connections of the mathematics with other research fields. For instance, the Pauli spin matrices used in Physics can be easily explained through quaternion analysis. Among the large number of developments in the last 150 years related to Hamilton's quaternion algebras, we could mention the algebra of octonions, discovered months later by Hamilton's friend J.T.Graves.

In this paper we will establish a De Moivre's extending relation for quaternions and octonions, in order to define a generalizing exponential function on both quaternions and octonions.

Let us consider for instance functions $f : H \longrightarrow H$ in the division of quaternions

$$f(u_1,u_2,u_3,u_1) \;=\; f_1(u_1,u_2,u_3,u_4) + \mathbf{i}\; f_2(u_1,u_2,u_3,u_4) \;+ \\ +\, \mathbf{j}\; f_3(u_1,u_2,u_3,u_4) + \mathbf{k}\; f_4(u_1,u_2,u_3,u_4),$$

having no restrictions over its coordinate functions $f_i : R^4$ ® R, except that they must be k times partially differentiable in their independent variables. In the Fueter theory of regular functions, the operator Γ is

$$\Gamma = \frac{\partial}{\partial u_1} + \mathbf{i}\frac{\partial}{\partial u_2} + \mathbf{j}\frac{\partial}{\partial u_3} + \mathbf{k}\frac{\partial}{\partial u_4}$$

With the multiplication rules given by

$$ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j,$$
$$i^2 = j^2 = k^2 = -1,$$
$$i1 = 1i = i,\ j1 = 1j = j,\ k1 = 1k = k,$$

the action of Γ over a quaternionic function f is given by,

$$\Gamma f = \left(\frac{\partial f_1}{\partial u_1} - \frac{\partial f_2}{\partial u_2} - \frac{\partial f_3}{\partial u_3} - \frac{\partial f_4}{\partial u_4}\right) + \mathbf{i}\left(\frac{\partial f_2}{\partial u_1} + \frac{\partial f_1}{\partial u_2} + \frac{\partial f_4}{\partial u_3} - \frac{\partial f_3}{\partial u_4}\right) + \\ + \mathbf{j}\left(\frac{\partial f_3}{\partial u_1} + \frac{\partial f_1}{\partial u_3} + \frac{\partial f_2}{\partial u_4} - \frac{\partial f_4}{\partial u_2}\right) + \mathbf{k}\left(\frac{\partial f_4}{\partial u_1} + \frac{\partial f_1}{\partial u_4} + \frac{\partial f_3}{\partial u_2} - \frac{\partial f_2}{\partial u_3}\right),$$

In a similar fashion, one defines a conjugation operator

$$\overline{\Gamma} = \frac{\partial}{\partial u_1} - \mathbf{i}\frac{\partial}{\partial u_1} - \mathbf{j}\frac{\partial}{\partial u_1} - \mathbf{k}\frac{\partial}{\partial u_1},$$

such that

$$\overline{\Gamma} f = \left(\frac{\partial f_1}{\partial u_1} + \frac{\partial f_2}{\partial u_2} + \frac{\partial f_3}{\partial u_3} + \frac{\partial f_4}{\partial u_4}\right) + \mathbf{i}\left(\frac{\partial f_2}{\partial u_1} - \frac{\partial f_1}{\partial u_2} - \frac{\partial f_4}{\partial u_3} + \frac{\partial f_3}{\partial u_4}\right) + \\ + \mathbf{j}\left(\frac{\partial f_3}{\partial u_1} + \frac{\partial f_2}{\partial u_2} - \frac{\partial f_1}{\partial u_3} - \frac{\partial f_2}{\partial u_4}\right) + \mathbf{k}\left(\frac{\partial f_4}{\partial u_1} - \frac{\partial f_3}{\partial u_2} + \frac{\partial f_2}{\partial u_3} - \frac{\partial f_1}{\partial u_4}\right),$$

Therefore, the following result holds:

$$Tf = \left(\frac{1}{2}\right)(\Gamma f + \overline{\Gamma} f) = \frac{\partial f_1}{\partial u_1} + \mathbf{i}\frac{\partial f_2}{\partial u_1} + \mathbf{j}\frac{\partial f_3}{\partial u1} + \mathbf{k}\frac{\partial f_4}{\partial u_1},$$

and it can be immediately verified, as expected, that the quaternion exponential

$$e^z = e^{u_1}\left\{\cos\left(\sqrt{u_2^2 + u_3^2 + u_4^2}\right) + \vec{u}\left(\frac{\operatorname{sen}\left(\sqrt{u_2^2 + u_3^2 + u_4^2}\right)}{\sqrt{u_2^2 + u_3^2 + u_4^2}}\right)\right\},$$

$$\vec{u} = u_2\mathbf{i} + u_3\mathbf{j} + u_4\mathbf{k} \qquad z = u_1 + \vec{u}$$

has the property

$$T(e^z) = e^z$$

For instance, one may define the operator

$$Sf = \left(\frac{1}{2}\right)\left(\Gamma f - \bar{\Gamma} f\right) = \left(-\frac{\partial f_2}{\partial u_2} - \frac{\partial f_3}{\partial u_3} - \frac{\partial f_4}{\partial u_4}\right) + \mathbf{i}\left(\frac{\partial f_1}{\partial u_2} + \frac{\partial f_4}{\partial u_3} - \frac{\partial f_3}{\partial u_4}\right) +$$

$$+\mathbf{j}\left(\frac{\partial f_1}{\partial u_3} + \frac{\partial f_2}{\partial u_4} - \frac{\partial f_4}{\partial u_2}\right) + \mathbf{k}\left(\frac{\partial f_1}{\partial u_4} + \frac{\partial f_3}{\partial u_2} - \frac{\partial f_2}{\partial u_3}\right),$$

and making $f = e^z$ one gets now the following relations:

$$\left(-\frac{\partial f_2}{\partial u_2} - \frac{\partial f_3}{\partial u_3} - \frac{\partial f_4}{\partial u_4}\right) = e^{u_1}\left(-\cos\left(\sqrt{u_2^2+u_3^2+u_4^2}\right) - \left(\frac{2\,\mathrm{sen}\left(\sqrt{u_2^2+u_3^2+u_4^2}\right)}{\sqrt{u_2^2+u_3^2+u_4^2}}\right)\right),$$

$$\mathbf{i}\left(-\frac{\partial f_1}{\partial u_2} + \frac{\partial f_4}{\partial u_3} - \frac{\partial f_3}{\partial u_4}\right) = -e^{u_1}\left(\frac{u_2\,\mathrm{sen}\left(\sqrt{u_2^2+u_3^2+u_4^2}\right)}{\sqrt{u_2^2+u_3^2+u_4^2}}\right)\mathbf{i},$$

$$\mathbf{j}\left(-\frac{\partial f_1}{\partial u_3} + \frac{\partial f_2}{\partial u_4} - \frac{\partial f_4}{\partial u_2}\right) = -e^{u_1}\left(\frac{u_3\,\mathrm{sen}\left(\sqrt{u_2^2+u_3^2+u_4^2}\right)}{\sqrt{u_2^2+u_3^2+u_4^2}}\right)\mathbf{j}$$

$$\mathbf{k}\left(\frac{\partial f_1}{\partial u_4} + \frac{\partial f_3}{\partial u_2} - \frac{\partial f_2}{\partial u_3}\right) = -e^{u_1}\left(\frac{u_4\,\mathrm{sen}\left(\sqrt{u_2^2+u_3^2+u_4^2}\right)}{\sqrt{u_2^2+u_3^2+u_4^2}}\right)\mathbf{k},$$

that leads to the conclusion:

$$S\,e^z = -\,e^z - A\,, \qquad A = e^{u_1}\left(\frac{2\,\mathrm{sen}\left(\sqrt{u_2^2+u_3^2+u_4^2}\right)}{\sqrt{u_2^2+u_3^2+u_4^2}}\right).$$

In other words, T leaves the quaternion exponential unchanged, while S gives the symmetric of e^z and translates the first coordinate of the quaternion exponential by a quantity A.

Analogously for octonions, being the octonionic exponential function given by

$$e^z = e^{u_1}e^{u_5}\left\{\cos\left(\sqrt{(u_2-u_6)^2+(u_3-u_7)^2+(u_4-u_8)^2}\right) + \right.$$

$$\left. + \vec{u}\left(\frac{\mathrm{sen}\left(\sqrt{(u_2-u_6)^2+(u_3-u_7)^2+(u_4-u_8)^2}\right)}{\sqrt{(u_2-u_6)^2+(u_3-u_7)^2+(u_4-u_8)^2}}\right)\right\}$$

$$\vec{u} = u_2\mathbf{i} + u_3\mathbf{j} + u_4\mathbf{k} - u_6\mathbf{i} - u_7\mathbf{j} - u_8\mathbf{k} \qquad z = u_1 - u_5 + \vec{u}\ ,$$

one can define the operator

$$Sf = \left(\frac{1}{2}\right)\left(\Gamma f - \bar{\Gamma} f\right) = \left(-\frac{\partial f_2}{\partial u_2} - \frac{\partial f_3}{\partial u_3} - \frac{\partial f_4}{\partial u_4} + \frac{\partial f_2}{\partial u_6} + \frac{\partial f_3}{\partial u_7} + \frac{\partial f_4}{\partial u_7}\right) +$$

$$+\mathbf{i}\left(\frac{\partial f_1}{\partial u_2} + \frac{\partial f_4}{\partial u_3} - \frac{\partial f_3}{\partial u_4} - \frac{\partial f_1}{\partial u_6} - \frac{\partial f_4}{\partial u_7} + \frac{\partial f_3}{\partial u_8}\right) +$$

$$+\mathbf{j}\left(-\frac{\partial f_4}{\partial u_2} + \frac{\partial f_1}{\partial u_3} + \frac{\partial f_2}{\partial u_4} + \frac{\partial f_4}{\partial u_6} - \frac{\partial f_1}{\partial u_7} - \frac{\partial f_2}{\partial u_8}\right) +$$

$$+\mathbf{k}\left(\frac{\partial f_3}{\partial u_2} - \frac{\partial f_2}{\partial u_3} + \frac{\partial f_1}{\partial u_4} - \frac{\partial f_3}{\partial u_6} - \frac{\partial f_2}{\partial u_7} - \frac{\partial f_1}{\partial u_8}\right) -$$

$$-\left(-\frac{\partial f_6}{\partial u_2} - \frac{\partial f_7}{\partial u_3} - \frac{\partial f_8}{\partial u_4} + \frac{\partial f_6}{\partial u_6} + \frac{\partial f_7}{\partial u_7} + \frac{\partial f_8}{\partial u_8}\right) -$$

$$-\mathbf{i}\left(\frac{\partial f_5}{\partial u_2} + \frac{\partial f_8}{\partial u_3} - \frac{\partial f_7}{\partial u_4} - \frac{\partial f_5}{\partial u_6} - \frac{\partial f_8}{\partial u_7} + \frac{\partial f_7}{\partial u_8}\right) -$$

$$-\mathbf{j}\left(-\frac{\partial f_8}{\partial u_2} + \frac{\partial f_5}{\partial u_3} + \frac{\partial f_6}{\partial u_4} + \frac{\partial f_8}{\partial u_6} - \frac{\partial f_5}{\partial u_7} - \frac{\partial f_6}{\partial u_8}\right) -$$

$$-\mathbf{k}\left(\frac{\partial f_7}{\partial u_2} - \frac{\partial f_6}{\partial u_3} + \frac{\partial f_5}{\partial u_4} - \frac{\partial f_7}{\partial u_6} + \frac{\partial f_6}{\partial u_7} - \frac{\partial f_5}{\partial u_8}\right)\ ,$$

and making $f = e^z$ one gets now the following relations:

$$\left(-\frac{\partial f_2}{\partial u_2} - \frac{\partial f_3}{\partial u_3} - \frac{\partial f_4}{\partial u_4} + \frac{\partial f_2}{\partial u_6} + \frac{\partial f_3}{\partial u_7} + \frac{\partial f_4}{\partial u_8}\right) =$$

$$= e^{u_1}e^{u_5}\left(-\cos\left(\sqrt{(u_2-u_6)^2+(u_3-u_7)^2+(u_4-u_8)^2}\right) - \right.$$

$$\left. -\left(\frac{-2\,\mathrm{sen}\left(\sqrt{(u_2-u_6)^2+(u_3-u_7)^2+(u_4-u_8)^2}\right)}{\sqrt{(u_2-u_6)^2+(u_3-u_7)^2+(u_4-u_8)^2}}\right)\right)$$

$$\mathbf{i}\left(\frac{\partial f_1}{\partial u_2} + \frac{\partial f_4}{\partial u_3} - \frac{\partial f_3}{\partial u_4} - \frac{\partial f_1}{\partial u_6} - \frac{\partial f_4}{\partial u_7} + \frac{\partial f_3}{\partial u_8}\right) =$$

$$= -2e^{u_1}e^{u_5}\left(\frac{u_2\,\mathrm{sen}\left(\sqrt{(u_2-u_6)^2+(u_3-u_7)^2+(u_4-u_8)^2}\right)}{\sqrt{(u_2-u_6)^2+(u_3-u_7)^2+(u_4-u_8)^2}}\right)\mathbf{i},$$

$$\mathbf{j}\left(-\frac{\partial f_4}{\partial u_2} + \frac{\partial f_1}{\partial u_3} + \frac{\partial f_2}{\partial u_4} + \frac{\partial f_4}{\partial u_6} - \frac{\partial f_1}{\partial u_7} - \frac{\partial f_2}{\partial u_8}\right) =$$

$$= -2e^{u_1}e^{u_5}\left(\frac{u_3\,\mathrm{sen}\left(\sqrt{(u_2-u_6)^2+(u_3-u_7)^2+(u_4-u_8)^2}\right)}{\sqrt{(u_2-u_6)^2+(u_3-u_7)^2+(u_4-u_8)^2}}\right)\mathbf{j}$$

$$\mathbf{k}\left(\frac{\partial f_3}{\partial u_2} - \frac{\partial f_2}{\partial u_3} + \frac{\partial f_1}{\partial u_4} - \frac{\partial f_3}{\partial u_6} - \frac{\partial f_2}{\partial u_7} - \frac{\partial f_1}{\partial u_8}\right) =$$

$$= -2e^{u_1}e^{u_5}\left(\frac{u_4\,\mathrm{sen}\left(\sqrt{(u_2-u_6)^2+(u_3-u_7)^2+(u_4-u_8)^2}\right)}{\sqrt{(u_2-u_6)^2+(u_3-u_7)^2+(u_4-u_8)^2}}\right)\mathbf{k}$$

$$\left(-\frac{\partial f_6}{\partial u_2}-\frac{\partial f_7}{\partial u_3}-\frac{\partial f_8}{\partial u_4}+\frac{\partial f_6}{\partial u_6}+\frac{\partial f_7}{\partial u_7}+\frac{\partial f_8}{\partial u_8}\right)=$$

$$= -\mathrm{e}^{\mathrm{u1}}\mathrm{e}^{\mathrm{u5}}\left(\cos\left(\sqrt{(u_2-u_6)^2+(u_3-u_7)^2+(u_4-u_8)^2}\right)+\right.$$

$$\left.+\left(\frac{2\,\mathrm{sen}\left(\sqrt{(u_2-u_6)^2+(u_3-u_7)^2+(u_4-u_8)^2}\right)}{\sqrt{(u_2-u_6)^2+(u_3-u_7)^2+(u_4-u_8)^2}}\right)\right)$$

$$-\,\mathbf{i}\left(\frac{\partial f_5}{\partial u_2}+\frac{\partial f_8}{\partial u_3}-\frac{\partial f_7}{\partial u_4}-\frac{\partial f_5}{\partial u_6}-\frac{\partial f_8}{\partial u_7}+\frac{\partial f_7}{\partial u_8}\right)=$$

$$=2\,\mathrm{e}^{\mathrm{u1}}\mathrm{e}^{\mathrm{u5}}\left(\frac{u_6\,\mathrm{sen}\left(\sqrt{(u_2-u_6)^2+(u_3-u_7)^2+(u_4-u_8)^2}\right)}{\sqrt{(u_2-u_6)^2+(u_3-u_7)^2+(u_4-u_8)^2}}\right)\mathbf{i}$$

$$-\,\mathbf{j}\left(-\frac{\partial f_8}{\partial u_2}+\frac{\partial f_5}{\partial u_3}+\frac{\partial f_6}{\partial u_4}+\frac{\partial f_8}{\partial u_6}-\frac{\partial f_5}{\partial u_7}-\frac{\partial f_6}{\partial u_8}\right)=$$

$$=2\,\mathrm{e}^{\mathrm{u1}}\mathrm{e}^{\mathrm{u5}}\left(\frac{u_7\,\mathrm{sen}\left(\sqrt{(u_2-u_6)^2+(u_3-u_7)^2+(u_4-u_8)^2}\right)}{\sqrt{(u_2-u_6)^2+(u_3-u_7)^2+(u_4-u_8)^2}}\right)\mathbf{j}$$

$$-\,\mathbf{k}\left(\frac{\partial f_7}{\partial u_2}-\frac{\partial f_6}{\partial u_3}+\frac{\partial f_5}{\partial u_4}-\frac{\partial f_7}{\partial u_6}+\frac{\partial f_6}{\partial u_7}-\frac{\partial f_5}{\partial u_8}\right)=$$

$$=2\,\mathrm{e}^{\mathrm{u1}}\mathrm{e}^{\mathrm{u5}}\left(\frac{u_8\,\mathrm{sen}\left(\sqrt{(u_2-u_6)^2+(u_3-u_7)^2+(u_4-u_8)^2}\right)}{\sqrt{(u_2-u_6)^2+(u_3-u_7)^2+(u_4-u_8)^2}}\right)\mathbf{k}$$

that leads to the conclusion:

$$S\,e^z = -\,e^z - A + B\,,$$

$$A = \mathrm{e}^{\mathrm{u1}}\mathrm{e}^{\mathrm{u5}}\left(\frac{2\,\mathrm{sen}\left(\sqrt{(u_2-u_6)^2+(u_3-u_7)^2+(u_4-u_8)^2}\right)}{\sqrt{(u_2-u_6)^2+(u_3-u_7)^2+(u_4-u_8)^2}}\right),$$

$$B = -\mathrm{e}^{\mathrm{u1}}\mathrm{e}^{\mathrm{u5}}\left(\frac{2\,\mathrm{sen}\left(\sqrt{(u_2-u_6)^2+(u_3-u_7)^2+(u_4-u_8)^2}\right)}{\sqrt{(u_2-u_6)^2+(u_3-u_7)^2+(u_4-u_8)^2}}\right)$$

In other words, T leaves the octonion exponential unchanged, while S gives the symmetric of e^z and translates the first coordinate of the quaternion exponential by a quantity $A + B$.

Conclusions

Quaternions and octonions may basically be considered as having their roots on extensions of the De Moivre's theorem. Connections between transcendent functions and operators of the Fueter type may then be well established.

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