On the Taylor's Series Expansion Theorem to prove the Barlow Points existence of Lagrange's Family Finite Element Interpolant

David Soares Pinto Júnior,
Depto. de Matemática, UFS,
49100-000, São Cristóvão, SE
E-mail: david@ufs.br.

Generally speaking, the superconvergence is defined as a higher order of convergence which is exhibited by interior points of the finite element solution where the derivative assumes an $O(h^{k+1})$ convergence, thus, one order higher when compared with an $O(h^k)$ of the finite element derivative field. Historycally, the higher order accuracy property was first noticed by Barlow[1] and its existence was firstly formulated by Strand and Fix[6]. This concept can be extended to similar properties but refering to $O(h^{k+2})$ convergence and $O(h^{k+3})$ convergence called simply ultraconvergence and hyperconvergence by using systematically the Taylor's Series Theorem[4]. The superconvergence is also linked to the concept of reduced integration and zero energy modes so important in the Finite Element Theory[8]. The superconvergence points are of utility as sampling points for the finite element designers in formulations like gradient smoothening or gradient recovery in the sense of The Superconvergent Patch Recovery as sketched by Zienkiewcz[7]. However, the major significance of these points that explains all kind of numerical and analytical studies in this topic refers to their application in adaptive finite element analysis, especially for construction of error indicators based on the recovered gradient or hessian.

Limiting the discussion to one dimensional problem for a discretization with equally spaced nodes and using a lagrangian basis, the finite element interpolant for a given exact function u is written simply as

$$u_h(x) = \sum_{i=0}^{k} u_i L_i(x), \tag{1}$$

where $u_i = u(x_i)$ is the degree of freedom calculated in the nodal point x_i. The functional degree of freedom u_i is, then, expanded around a point $\overline{x}$ inside the element by applying the Taylor's series expansion which can be expressed as

$$u(x_i) = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{d^i}{dx^i} u(\overline{x}) h_x^i, \tag{2}$$

where $h_x = x_i - \overline{x}$. In order to characterize the existence of superconvergent points for the first order derivative the main idea consists of calculating the points $\overline{x}$ inside the element where the exact error expression satisfy the superconvergence condition given by $||u'(\overline{x}) - u'_h(\overline{x})|| = O(h^{k+1})$.

Hence, starting from a Lagrange's Family element interpolant which derivative is calculated by differentiating the interpolant, a set of superconvergent points for the first order derivative is, then, obtained by calculating the zeros of the following polynomial:

$$s(x) = \sum_{i=0}^{k} \frac{dL_i(x)}{dx} x_i^{k+1} - (k+1)x^k = 0, \tag{3}$$

where $\{L_i\}_{i=0}^{k}$ is the Lagrange's Basis, k is the spectral order of the interpolation and x is the coordinate of an arbitrary point inside the reference element. If the symbolic software, *Mathematica*, is used a remarkable set of points can be calculated analytically which are the zeros of Legendre polynomials and, hence, the Gaussian quadrature points but not for higher spectral order as infered from Table 1. It's important to emphasize that the completeness property of the Lagrange's basis is fundamental in order to simplify the deduction by applying the ideas firstly discussed by Carey[2]. Basically, the completeness consists of several identities that depends on the degree of lagragian basis function. The fundamental expression in terms of completeness property represents the partition of unity and is written as

$$\sum_{i=0}^{k} L_i(x) = 1. \tag{4}$$

Similarly, expressions like the previous one can be proved but their number restricts their citation.

Let us consider a three node one dimensional element. Assuming that the midpoint x_1 is free the superconvergence function $s(x)$ is written as

$$s(x) = a_2(x_1)x^2 + a_1(x_1)x + a_0(x_1) = 0. \qquad (5)$$

If the nodal points are equally spaced inside the element this polynomial coincides with the Legendre polynomial where $a_2(x_1) = 3, a_1(x_1) = 0, a_0(x_1) = -1$, which zeros are the Gaussian quadrature points. However, if the midpoint is randomlly positioned inside the element, then, the superconvergence function is

$$s(x) = 3x^2 + 2x_1x - 1 = 0. \qquad (6)$$

Therefore, the polynomial coefficients are not invariant under a coordinate changing of the nodal points. Such counter-example shows clearly that the superconvergent points position may vary inside the element for an arbitrary element with randomlly distributed nodes.

The consistence of formulas can be also applied to explain the coincidence between the Barlow points and Gaussian quadrature points for linear elements in 1D with nodes x_0 and x_1. In this particular case, the superconverence points is simply the zero of the superconvergence function

$$s(x) = -2x + x_0 + x_1 = 0. \qquad (7)$$

Obviously, the zero is the centroid of the element which coincides exactly with the Gauss quadrature point obtained by calculating the zero of the Legendre's polynomial over the reference element $[-1, +1]$. One can reproduce exactly the superconvergence function for a two node element by comparing the former with the Legendre's polynomial written in the form

$$P_n(x) = D_x^n[(x^2 - 1)^n], D_x^n = \frac{d^n}{dx^n}, \qquad (8)$$

for $n = 0$ and determining the zeros of $P_{n+1}(x) = P_{0+1}(x) = D_x^1[(x^2 - 1)^1]$.

Let us now assume that the nodal configuration inside a five nodes one dimensional element permits x_1 and x_3 vary freely while $x_0 = -1$,$x_2 = 0$ and $x_4 = +1$. To simplify the algebrism, is assumed that x_1 and x_3 are symmetrically placed, thus, $x_1 = -x_3$. After substituting all completeness properties for a quartic element and the symmetry condition imposed, the superconvergence funtion is written as

$$s(x) = 5x^4 + [-3(1 + x_1^2)]x^2 + x_1^2 = 0. \qquad (9)$$

Firstly, this confirms one more time that the coefficients depend on, in general, the nodal coordinates. In conclusion, the superconvergence function are not invariant for a configuration of randomlly distributed nodes inside the element.

Similar calculations can be performed for a four nodes one dimensional element by assuming, without limitation of generality, that x_1 and x_2 are varying freely under the symmetric configuration of nodes expressed as $x_1 = -x_2$ while the end nodes are fixed and set to be $x_0 = -1$ and $x_3 = +1$. In this case, the superconvergence function is written in the form

$$s(x) = 2x^3 - (1 + x_1^2)x = 0. \qquad (10)$$

By comparing the superconvergence function $s(x)$ for a cubic element with a Legendre's polynomial of third degree which zeros are calculated from $5x^3 - 3x = 0$, one can infer that the zeros of $s(x)$ reproduces exactly the Barlow points, setting $x_1 = -x_2 = -1/3$ as illustrated in Table 1.

In fact, MacNeal[3] found out that Gaussian quadrature points and Barlow points are coincident for lower order elements whereas it is not true for higher order elements as illustrated in Table 1. Similarly, Prathap[5] showed how to determine the Barlow points from a variational basis and from the Taylor's Series approach is also possible to work out that for not equally spaced nodes inside the element, in general, the superconvergence point can change its coordinate.

In this direction, one open question emerges from the invertibility of the isoparametric mapping related to the existence of superconvergent points if distorced and higher order elements are used. The natural question is now directed to the geometric transformation under which the reference element is mapped onto an arbitrary, general element. Mathematically, the problem consists of calculating the zeros of superconvergence function $s(F(\xi))$, where $x = F(\xi)$ represents a geometric mapping between the cartesian co-ordinates system x and the normalized co-ordinates system ξ. What assumptions should be imposed to the geometric mapping $x = F(\xi)$ in order to keep on the superconvergent point? In finite element calculations, the mapping $x = F(\xi)$ is the well-known isoparametric mapping that is not a afinne mapping for higher order elements. In summary, what such a kind of geometric transformations can preserve the set of superconvergent points inside the element? It is a particularly important point since that in finite element applications the mesh refinement leads to varying valence discretizations and several error indicators are based on gradient recovery. For higher spatial dimensions the complexity is seriously higher since that the set of superconvergent points represent a superconvergence curve in 2D case or a superconvergence surface in 3D case. These topics will be discussed as well.

k	Barlow Points	Gauss Points
1	0	0
2	$\pm\sqrt{3}/3$	$\pm\sqrt{3}/3$
3	$0, \pm\sqrt{5}/3$	$0, \pm\sqrt{3/5}$
4	$\pm\frac{\sqrt{3\pm\sqrt{29/5}}}{2\sqrt{2}}$	$\pm\sqrt{\frac{3}{7}\pm\frac{2\sqrt{6}}{7\sqrt{5}}}$

Tabela 1: Spectral order and coordinates for superconvergent points in the sense of Taylor's expansion in 1D.

Referências

[1] J. Barlow, Optimal stress locations in finite element models, *Int. J. for Num. Meth. in Engng.*, 10 (1976) 243-251.

[2] R.G. Mackinnon, G.F. Carey, Superconvergent derivatives: A Taylor Series Analysis, *Int. J. for Num. Meth. in Engng.*, 28 (1989)489-509.

[3] R.H. MacNeal, Finite Elements: Their Design and Performance, *Marcel Dekker*, New York (1994).

[4] D.S. Pinto Jr., On Higher Order Accuracy Points for First and Second Order Derivatives of Lagrangian Finite Element Interpolants, XXV CNMAC, Nova Friburgo, RJ (2002).

[5] G. Prathap, Barlow Points and Gauss Points and the aliasing and best fit paradigms, *Computers and Structures*, 58 (1996)321-325.

[6] G. Strang, GJ. Fix, An analysis of the finite element method, *Prentice-Hall:Englewood Cliffs*, NJ (1973).

[7] O.C. Zienkiewicz, JZ. Zhu, The Superconvergent patch recovery and a posteriori error estimates. Part 1: The recovery technique, *Int. J. for Num. Meth. in Engng.*, 33(1992) 1331-1364.

[8] M.Zlamal, Superconvergence and reduced integration in the finite element method, *Math. Computations*, 32(1978) 663-685.