Spectral Analysis of the Matrix for Three-dimensional Discrete Ordinates Problems

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Based on a spectral analysis of the $\mathbf{LTS_N}$ matrices, we describe a new algorithm to generate an analytical solution for discrete ordinates problems in three-dimensional cartesian geometry with isotropic scattering and one energy group that appear as:

$$\begin{gathered}\left[\mu_m \frac{\partial}{\partial x} + \eta_m \frac{\partial}{\partial y} + \xi_m \frac{\partial}{\partial z} + \sigma_t\right] \Psi_m(x,y,z) = \\ Q(x,y,z) + \frac{\sigma_s}{8} \sum_{n=1}^{M} w_n \Psi_n(x,y,z)\,,\end{gathered} \tag{1}$$

where $m = 1 : M$ are positive integer numbers, $M = N(N+2)$ is the cardinality of the discrete ordinates set, N represents the order of the angular quadrature, σ_t is the macroscopic total cross-section, σ_s is the diferential scattering cross-section, $Q(x,y,z)$ the source term , $\Psi_m(x,y,z)$ is the angular flux in the discrete direction $\Omega_m = (\mu_m, \eta_m, \xi_m)$ and w_m is the weight of the angular quadrature set used to model the transport problem (Lewis [9]).

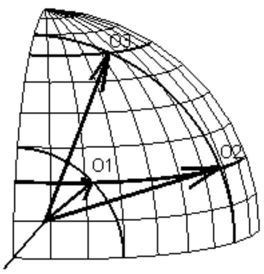

Figure 1: $\Omega_m = (\mu_m, \eta_m, \xi_m), m = 1 : \frac{M}{8}, N = 4$

Furthemore, we first transverse integrate the S_N equations and then we apply the Laplace transform to generate the systems of linear algebric equations, that is

$$\begin{aligned}(sI - A_x)\overline{\Psi}_x(s) &= \Psi_x(0) + \overline{S}_x(s),\\ (sI - A_y)\overline{\Psi}_y(s) &= \Psi_y(0) + \overline{S}_y(s),\\ (sI - A_z)\overline{\Psi}_z(s) &= \Psi_z(0) + \overline{S}_z(s).\end{aligned} \tag{2}$$

The solutions of the linear systems (2) are given by:

$$\begin{aligned}\overline{\Psi}_x(s) &= (sI - A_x)^{-1}[\Psi_x(0) + \overline{S}_x(s)],\\ \overline{\Psi}_y(s) &= (sI - A_y)^{-1}[\Psi_y(0) + \overline{S}_y(s)],\\ \overline{\Psi}_z(s) &= (sI - A_z)^{-1}[\Psi_z(0) + \overline{S}_z(s)].\end{aligned} \tag{3}$$

The entries of the matrix A_z, with $i = 1 : M$ and $j = 1 : M$, have the form

$$a_z(i,j) = \left\{\begin{array}{cc} -\dfrac{8\sigma_t - \sigma_{sii} w_i}{8\xi_i} & se \quad i = j \\ & \\ \dfrac{\sigma_{sij} w_j}{8\xi_i} & se \quad i \neq j \end{array}\right. \,, \tag{4}$$

$$\Psi_z(0) = \left[\ \Psi_{1z}(0)\ \Psi_{2z}(0) \cdots \Psi_{Mx}(0)\ \right]^T, \tag{5}$$

the vector of unknowns is

$$\overline{\Psi}_z(s) = \left[\ \overline{\Psi}_{1z}(s)\ \overline{\Psi}_{2z}(s) \cdots \overline{\Psi}_{Mz}(s)\ \right]^T \tag{6}$$

and the vector $S_z(s)$ has generic components

$$\begin{aligned}\overline{S}_{zi}(s) = \tfrac{\overline{Q}_z(s)}{\xi_i} - \tfrac{\mu_i}{a\xi_m}\left[\overline{\Psi}_i^x(0,s) - \overline{\Psi}_i^x(a,s)\right]\\ - \frac{\eta_i}{b\xi_i}\left[\overline{\Psi}_i^y(0,s) - \overline{\Psi}_i^y(b,s)\right].\end{aligned} \tag{7}$$

In this work we describe the diagonalization

$$A_z = V_z D_z V_z^{(-1)}, \tag{8}$$

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where, D_z is the diagonal matrix of eigenvalues of A_z and V_z the matrix of the respective eigenvectors. Similarly, $A_y = V_y D_y V_y^{(-1)}$ and $A_x = V_x D_x V_x^{(-1)}$. This is possible from the complete description of the spectra of these matrices, because any eigenvalues are multiples(Tabel 1) but the respectives eigenvectors are independet lineary.

N	$\frac{\sigma_t}{\xi_1}$	$\frac{\sigma_t}{\xi_2}$	$\frac{\sigma_t}{\xi_3}$	$\frac{\sigma_t}{\xi_4}$	$\frac{\sigma_t}{\xi_5}$	$\frac{\sigma_t}{\xi_6}$	$\frac{\sigma_t}{\xi_7}$	$\frac{\sigma_t}{\xi_8}$
2	3	-	-	-	-	-	-	-
4	7	3	-	-	-	-	-	-
6	11	7	3	-	-	-	-	-
8	15	11	7	3	-	-	-	-
10	19	15	11	7	3	-	-	-
12	23	19	15	11	7	3	-	-
14	27	23	19	15	11	7	3	-
16	31	27	23	19	15	11	7	3

Table 1: Multiples eigenvalues of A_z for $\Omega_m = (\mu_m, \eta_m, \xi_m)$,$m = 1 : \frac{M}{2}$

We assume a solution in the exponential form for the homogeneous equations associate with the one-dimensional transverse integrated $\mathbf{S_N}$ nodal equations. Therefore we obtain the eigenvalue problem

$$\begin{aligned} \alpha_m(s)(\sigma_t + s\mu_m) &= \frac{\sigma_s}{8}\sum_{n=1}^{M}\alpha_n w_n, \\ \alpha_m(s)(\sigma_t + s\eta_m) &= \frac{\sigma_s}{8}\sum_{n=1}^{M}\alpha_n w_n, \\ \alpha_m(s)(\sigma_t + s\xi_m) &= \frac{\sigma_s}{8}\sum_{n=1}^{M}\alpha_n w_n. \end{aligned} \tag{9}$$

In order to determine the angular fluxes we apply the inverse Laplace transform to (3) $\pounds^{-1}[(sI - A_z)^{-1}] = \pounds^{-1}[\big(sV_zV_z^{-1} - V_zD_zV_z^{-1}\big)^{-1}]$

$$= \mathrm{V}_z\pounds^{-1}[(sI - D_z)^{-1}]V_z^{-1} = V_z e^{D_z z}V_z^{-1} \ .$$

In [8] was determined two algorithms whose difference lies in the representation of the transverse leakage terms that appear in the transverse integrated S_N two-dimensional equations. These terms in version $LTS_N2D-Diag$ are written as linear combinations of the eigenvectors multiplied by exponential functions of the corresponding eigenvalues. As with in the $LTS_N2D - DiagExp$ method, the transverse leakage terms are represented by exponential functions with decay constant depending on the characteristics of the material associated to the medium.

We proceed in a similar form with three-dimensional problem and representing the convolution operation by $*$, we determine the analytical matrix form for the edge-average angular flux as function of the average angular flux at the bondaries.

$$\Psi_x(x) = [V_x e^{D_x x}V_x^{-1}]\Psi_x(0) + [V_x e^{D_x x}V_x^{-1}] * S_x(x)\,,$$

$$\Psi_y(y) = [V_y e_y^{D} y V_y^{-1}]\Psi_y(0) + [V_y e^{D_y y}V_y^{-1}] * S_y(y)\,,$$

$$\Psi_z(z) = [V_z e^{D_z z}V_z^{-1}]\Psi_z(0) + [V_z e^{D_z z}V_z^{-1}] * S_z(z)\,.$$

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