An explicit Jordan Decomposition of Companion matrices

Fermín S. V. Bazán
Departamento de Matemática, CFM, UFSC
88040-900, Florianópolis, SC
E-mail: fermin@mtm.ufsc.br

S. Gratton
CERFACS, 42 Av. Gaspard Coriolis
31057, Toulouse Cedex 01. France
E-mail: gratton@cerfacs.fr.

Abstract

We derive a closed form for the Jordan decomposition of companion matrices with emphasis on properties of generalized eigenvectors. As a by product we provide a formula for the inverse of confluent Vandermonde matrices and results on sensitivity of roots of polynomials.

1 Introduction

We are concerned with companion matrices of the form

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{m-1} \end{bmatrix} \quad (1)$$

where $a_i \in \mathbb{C}, i = 0, \ldots, m-1$. Matrices like this appear in a variety of areas in science and engineering [1, 2, 3, 5, 9, 11]. There is a close relationship between companion matrices and polynomials in a complex variable. Of course, roots of polynomials can be computed as eigenvalues of C and vice versa. This relies on the fact that the characteristic polynomial of C, $\det(tI - C) = \pi(t)$, is readily proved to be $\pi(t) = a_0 + a_1 t + \cdots + a_{m-1}t^{m-1} + t^m$.

We denote by $\lambda_1, \ldots, \lambda_p$ the p distinct eigenvalues of C and by $m_1, \ldots, m_p$ their respective algebraic multiplicities (i.e, $\pi(t) = (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2}\cdots(t-\lambda_p)^{m_p}$ with $m_1 + \cdots + m_p = m$). The fact that C is a non derogatory matrix [8, 10] ensures thus that a particular Jordan decomposition of C can be written as

$$\begin{bmatrix} J_{\lambda_1} & & \\ & \ddots & \\ & & J_{\lambda_p} \end{bmatrix} = \begin{bmatrix} L_1 \\ \vdots \\ L_p \end{bmatrix} C \begin{bmatrix} R_1 & \ldots & R_p \end{bmatrix},$$

where for $i = 1, \ldots p$,

$$J_{\lambda_i} = \begin{pmatrix} \lambda_i & 1 & & & \\ & . & . & & \\ & & . & . & \\ & & & . & 1 \\ & & & & \lambda_i \end{pmatrix} \in \mathbb{C}^{m_i \times m_i},$$

$$\begin{bmatrix} L_1 \\ \vdots \\ L_p \end{bmatrix} \begin{bmatrix} R_1 & \ldots & R_p \end{bmatrix} = I \quad (2)$$

where I is the $m \times m$ identity matrix, $R_i \in \mathbb{C}^{m \times m_i}$, and $L_i \in \mathbb{C}^{m_i \times m}$. The columns of R_i (resp. L_i^*) represent a right (resp. left) Jordan chain associated with λ_i the leading eigenvector being $R_i e_1^{[m_i]}$ (resp. $L^* e_{m_i}^{[m_i]}$). The star symbol denotes conjugate transpose, i.e., $L^* = \bar{L}^T$, and $e_j^{[m_i]}$ is the j-th column of the $m_i \times m_i$ identity matrix.

In this work we describe a closed form for the Jordan decomposition of C concentrating on properties of generalized eigenvectors. This leads to a formula for the inverse of Confluent Vandermonde matrices and results on the sensitivity of the roots of $\pi(t)$.

2 Jordan Decomposition of Companion Matrices

In this section, we provide an explicit Jordan decomposition of C. The following technical result will be needed.

Lemma 2.1 *For arbitrary $\lambda \in \mathbb{C}$ we set $\phi(\lambda) = [1, \lambda, \ldots, \lambda^{m-1}]^T$ and define by $\phi^{(m)}(\lambda)$ the m-th derivative of $\phi(\lambda)$ with respect to λ.*

Let H be the $m \times m$ matrix

$$H = \begin{bmatrix} a_1 & a_2 & \cdots & a_{m-1} & 1 \\ a_2 & \cdots & a_{m-1} & 1 & \\ \vdots & \vdots & 1 & & \\ a_{m-1} & 1 & & & \\ 1 & & & & \end{bmatrix}. \quad (3)$$

Then for any integers i and j, there holds

$$\frac{\phi^{(i)T}(\lambda)}{i!} H \frac{\phi^{(j)}(\lambda)}{j!} = \frac{\pi^{(i+j+1)}(\lambda)}{(i+j+1)!}.$$

Proof. The proof is done by induction on i. For $i = 0$ an elementary computation shows that

$$\phi^T(\lambda) H \frac{\phi^{(j)}(\lambda)}{j!} = \frac{\pi^{(j+1)}(\lambda)}{(j+1)!}.$$

Assume now that for a given i,

$$\frac{\phi^{(i)T}(\lambda)}{i!} H \frac{\phi^{(j)}(\lambda)}{j!} = \frac{\pi^{(i+j+1)}(\lambda)}{(i+j+1)!}.$$

Taking the derivative of the above expression with respect to λ yields

$$\frac{\phi^{(i+1)T}(\lambda)}{i!} H \frac{\phi^{(j)}(\lambda)}{j!} + \frac{\phi^{(i)T}(\lambda)}{i!} H \frac{\phi^{(j+1)}(\lambda)}{j!} = \frac{\phi^{(i+1)T}(\lambda)}{i!} H \frac{\phi^{(j)}(\lambda)}{j!} + (j+1)\frac{\pi^{(i+j+2)}(\lambda)}{(i+j+2)!} = \frac{\pi^{(i+j+2)}(\lambda)}{(i+j+1)!},$$

which shows that

$$\frac{\phi^{(i+1)T}(\lambda)}{i!} H \frac{\phi^{(j)}(\lambda)}{j!} = \frac{\pi^{(i+j+2)}(\lambda)}{(i+j+1)!} - (j+1)\frac{\pi^{(i+j+2)}(\lambda)}{(i+j+2)!} = \frac{(i+1)\pi^{(i+j+2)}(\lambda)}{(i+j+2)!}$$

which completes the proof. □

Proposition 2.2 *Define $R_l = [r_1, r_2, \cdots, r_{m_l}]$ where $r_i = H \frac{\phi^{(i-1)}(\lambda_l)}{(i-1)!}$. The set $\{r_1, r_2, \cdots, r_{m_l}\}$ is a right Jordan chain of C associated with the eigenvalue λ_l and r_1 is the leading right eigenvector. Similarly, define $\breve{L}_l^* = [\breve{l}_1, \breve{l}_2, \cdots, \breve{l}_{m_l}]$ where $\breve{l}_i = \frac{\bar{\phi}^{(m_l-i)}(\lambda_l)}{(m_l-i)!}$. The set $\{\breve{l}_1, \breve{l}_2, \cdots, \breve{l}_{m_l}\}$ is a left Jordan chain of C associated with the eigenvalue λ_l and $\breve{l}_{m_l}$ is the leading left eigenvector. The left and right generalized Jordan chains satisfy*

$$\breve{L}_l R_l \equiv \begin{bmatrix} \breve{l}_1^* \\ \vdots \\ \breve{l}_{m_l}^* \end{bmatrix} \begin{bmatrix} r_1 & \ldots & r_{m_l} \end{bmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 & . & \alpha_{m_{l-1}} & \alpha_{m_l} \\ & . & . & . & \alpha_{m_{l-1}} \\ & & . & . & . \\ & & & . & \alpha_2 \\ & & & & \alpha_1 \end{pmatrix} \equiv F_l \in \mathbb{C}^{m_l \times m_l}, \quad (4)$$

where $\alpha_i = \dfrac{\pi^{(m_l+i-1)}(\lambda_l)}{(m_l+i-1)!}$.

Proof. For arbitrary λ of multiplicity q consider the vectors $r_1, \ldots, r_q$. It is clear that these vectors are linearly independent. Thus, if we set $r_0 = 0$, we have to prove that r_1 is a right eigenvector of C associated with λ and that

$$(C - \lambda I) r_j = r_{j-1}, \quad 1 \le j \le q. \quad (5)$$

For this, if $x = [x_1 \cdots x_m]^T$ is a right eigenvector of C associated with λ then

$$Cx = \lambda x \Longleftrightarrow \begin{cases} -a_0 x_m & = \lambda x_1 \\ x_1 - a_1 x_m & = \lambda x_2 \\ \quad \vdots & \\ x_{m-2} - a_{m-2} x_m & = \lambda x_{m-1} \\ x_{m-1} - a_{m-1} x_m & = \lambda x_m \end{cases}$$

This shows that x_m cannot be zero otherwise x would be the 0 vector. Setting $x_m = 1$ it is easy to see $x = H\phi(\lambda)$. Thus one has

$$CH\phi(\lambda) = \lambda H \phi(\lambda). \quad (6)$$

We now prove the conditions (5). Taking derivative with respect to λ in (6) we have

$$CH\phi^{(1)}(\lambda) = H\phi(\lambda) + \lambda H \phi^{(1)}(\lambda). \quad (7)$$

This shows that (5) holds for $j = 2$, and an inductive argument obtained by repeated differentiation of (7) concludes the proof in the case of the right generalized eigenvectors. A similar proof can be obtained for the generalized left eigenvectors by starting with $\phi(\lambda)^T C^T = \phi(\lambda)^T \lambda$ instead of (6) and taking the derivatives of this equality. The normalization factors α_i are a consequence of Lemma 2.1. □

To obtain the Jordan decomposition, we transform the left Jordan chain so that the normalization (2) holds.

Proposition 2.3 *Define $L_l^* = [l_1, l_2, \cdots, l_{m_l}] = \breve{L}_l^* F_l^{-*}$. The set $\{l_1, l_2, \cdots, l_{m_l}\}$ is a left Jordan chain of C associated with the eigenvalue λ_l, l_{m_l} being the leading left eigenvector.*
The left and right generalized Jordan chains are normalized so that

$$L_l R_l \equiv \begin{bmatrix} l_1^* \\ \vdots \\ l_{m_l}^* \end{bmatrix} \begin{bmatrix} r_1 & \ldots & r_{m_l} \end{bmatrix} = I \in \mathbb{R}^{m_l \times m_l}. \quad (8)$$

Similarly, we define $\breve{R}_l = [\breve{r}_1, \breve{r}_2, \ldots, \breve{r}_{m_l}] = [r_1, \ldots r_{m_l}] F_l^{-1}$. The set $\{\breve{r}_1, \breve{r}_2, \ldots, \breve{r}_{m_l}\}$ is a right Jordan chain of C associated with the eigenvalue λ_l, and $\breve{L}_l \breve{R}_l = I \in \mathbb{R}^{m_l \times m_l}$.

Proof. Let γ_i be defined by the recursion

$$\begin{array}{l}\gamma_1 = 1/\alpha_1, \\ \gamma_{i+1} = -\frac{1}{\alpha_1}\sum_{k=1}^{i} \alpha_{i-k+2}\gamma_k, \quad i = 1, \ldots m_l - 1.\end{array},$$

in such a way that

$$G = \begin{pmatrix} \gamma_1 & \gamma_2 & \cdot & \gamma_{m_l-1} & \gamma_{m_l} \\ & \cdot & \cdot & \cdot & \gamma_{m_l-1} \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \gamma_2 \\ & & & & \gamma_1 \end{pmatrix} = F_l^{-1}$$

The set $\{\bar{\breve{l}}_{m_l} \ldots \bar{\breve{l}}_1\}$ forms a right Jordan chain of C^T associated with λ_l. For any nonsingular matrix X commuting with J_{λ_l}, $[\bar{\breve{l}}_{m_l} \ldots \bar{\breve{l}}_1]X$ is a right Jordan chain C^T associated with λ_l. By definition of the l_i's, $\left[\begin{array}{ccc} l_{m_l} & \ldots & l_1 \end{array}\right] = \left[\begin{array}{ccc} \breve{l}_{m_l} & \ldots & \breve{l}_1 \end{array}\right]\bar{G}$. A direct computation shows that G commutes with J_{λ_l}, which implies that $\{\bar{l}_{m_l}, \cdots, \bar{l}_1\}$ is a right Jordan chain of C^T associated with λ_l, that is, $\{l_l, \ldots, l_{m_l}\}$ is a left Jordan chain of C associated with λ_l. Since by definition of the γ_i's, $GF_l = F_lG = I$, it follows

$$\begin{bmatrix} l_1^* \\ \vdots \\ l_{m_l}^* \end{bmatrix} [r_1 \ldots r_{m_l}] = G \begin{bmatrix} \breve{l}_1^* \\ \vdots \\ \breve{l}_{m_l}^* \end{bmatrix} [r_1 \ldots r_{m_l}] = I,$$

and the first part of the proposition is proved. The proof of the remaining part is a consequence of Eq. (4) since $[r_1 \ldots r_{m_l}]F_l^{-1}$ is a right Jordan chain of C associated with λ_l as we have seen that F^{-1} commutes with J_{λ_l}. $\square$

3 Eigenvector properties

An immediate consequence of (8) is an explicit formula for computing the inverse of confluent Vandermonde matrices as described in the corollary below.

Corollary 3.1 (Inversion formula) *Let $\breve{L}$ be the confluent Vandermonde matrix defined by $\breve{L}^* = [\breve{L}_1^* \ldots \breve{L}_p^*]$. Then $\breve{L}^{-1} = [R_1 \quad \cdots R_p]F^{-1}$ with $F = \mathrm{diag}(F_1, \ldots, F_p)$.*

It is known that right eigenvectors of companion matrices like the one we use here can be computed by applying the Euclidean algorithm to divide $\pi(t)$ by $(t - \lambda_l)$ (see, e.g., Toh and Trefethen [6] or Bezerra and Bazán [3]) yielding in our notations a deflated polynomial

$$\pi_1(t) \equiv \phi(t)^T r_1 = \pi(t)/(t - \lambda_l).$$

This shows that the coefficients of the right eigenvector can be seen as coefficients of a deflated polynomial. The following proposition shows that the complete right Jordan chain $[r_1 \ldots r_{m_l}]$ also enjoy this property.

Proposition 3.2 *Define $\pi_i(t) = \phi(t)^T r_i$ $(i = 1, \ldots, m_l)$ where r_i are generalized right eigenvectors of C as introduced in Prop. 2.2. Then π_i is a monic polynomial of degree $m - i$ of the form*

$$\pi_i(t) = (t - \lambda_l)^{m_l - i} \prod_{\substack{j=1 \\ j \neq l}}^{p} (t - \lambda_j)^{m_j}. \tag{9}$$

Proof: It is clear that all π_i are monic polynomials of degree $m - i$. The definition of the r_i's and successive differentiation imply

$$\begin{array}{rcl} \pi_1(t) & = & \phi^T(t)\, H\phi(\lambda_l) \\ \pi_1^{(1)}(t) & = & \phi^{(1)^T}(t)\, H\phi(\lambda_l) \\ \vdots & & \\ \pi_1^{(i)}(t) & = & \phi^{(i)^T}(t)\, H\phi(\lambda_l) \\ \vdots & & \\ \pi_1^{(m_l-1)}(t) & = & \phi^{(m_l-1)^T}(t)\, H\phi(\lambda_l). \end{array} \tag{10}$$

If $t = \lambda_l$, Prop. 2.1 implies that (10) can be rewritten as

$$\begin{array}{rcl} \pi_i(\lambda_l) & = & \pi_1^{(i-1)}(\lambda_l) = (i-1)!\phi^T(\lambda_l)H\frac{\phi^{(i-1)}(\lambda_l)}{(i-1)!} \\ & = & (i-1)!\pi^{i-1}(\lambda_l) \quad i = 1, \ldots, m_l \end{array}$$

But since λ_l is a multiple root of π, this equality implies that λ_l is a root of π_i $(i = 1 \ldots m_l - 1)$ and a recursive argument shows that this root is of multiplicity $m_l - i$. If $t = \lambda_k \neq \lambda_l$, a similar procedure and the existing biorthogonality condition between left and right generalized eigenvectors leads to

$$\begin{array}{rl} \pi_i(\lambda_k) & = \pi_1^{(i-1)}(\lambda_k) \\ & = (i-1)!\phi^T(\lambda_l)H\frac{\phi^{(i-1)}(\lambda_k)}{(i-1)!} \\ & = 0, \quad i = 1, \ldots, m_l, \end{array}$$

which concludes the proof. $\square$

Remark A comment concerning the meaning of this proposition is in order. Let C_i $(i = 1, \ldots, m_l - 1)$ denote the $(m - i) \times (m - i)$ companion matrix associated with the polynomial π_i and, for $i = 1, \ldots, m_l$, let $\check{r}_i$ be the vector formed by taking the first $m - i + 1$ components of r_i. Then, with the convention that $C_0 = C$, the proposition ensures that

$$C_{i-1}\check{r}_i = \lambda_l \check{r}_i, \quad i = 1, \ldots m_l, \tag{11}$$

and λ_l is a simple eigenvalue of the companion matrix C_{m_l-1}. For future reference, the left eigenvector of C_{m_l-1}, will be denoted by $\psi(\lambda_l)$. It is defined by

$$\psi(\lambda_l) = [1, \bar{\lambda}_l, \cdots, \bar{\lambda}_l^{m-m_l}]^T. \tag{12}$$

3.1 Numerical illustration: Jordan decomposition

We present an illustration of the above notions for $m = 5$, $(\lambda_1, m_1) = (1, 2)$, $(\lambda_2, m_2) = (2, 2)$,

$(\lambda_3, m_3) = (3,1)$, in which case, $\pi(t) = (t-1)^2(t-2)^2(t-3)$. We show how to obtain easily a Jordan form of the companion matrix associated with π. Note that $\pi(t) = t^5 - 9t^4 + 31t^3 - 51t^2 + 40t - 12$.

- Case of $\lambda = 1$, $m_1 = 2$.
  From $(t-2)^2(t-3) = t^3 - 7t^2 + 16t - 12$ and $(t-1)(t-2)^2(t-3) = t^4 - 8t^3 + 23t^2 - 28t + 12$ follows using Proposition 3.2 and the definition of the $\breve{l}_i$'s that

$$R_1 = \begin{pmatrix} 12 & -12 \\ -28 & 16 \\ 23 & -7 \\ -8 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \breve{L}_1^* = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix}.$$

  From $\pi^{(2)}(1)/2 = -2$ and $\pi^{(3)}(1)/6 = 5$, we obtain

$$F_1 = \begin{pmatrix} -2 & 5 \\ 0 & -2 \end{pmatrix}, \; F_1^{-1} = \frac{1}{4}\begin{pmatrix} -2 & -5 \\ 0 & -2 \end{pmatrix}$$

$$\text{and } L_1^* = \breve{L}_1^* F_1^{-*} = \tfrac{1}{4}\begin{pmatrix} -5 & -2 \\ -7 & -2 \\ -9 & -2 \\ -11 & -2 \\ -13 & -2 \end{pmatrix}.$$

- The same calculation for the two remaining roots gives

$$\begin{aligned} R &= [R_1, R_2, R_3] \\ &= \begin{pmatrix} 12 & -12 & 6 & -3 & 4 \\ -28 & 16 & -17 & 7 & -12 \\ 23 & -7 & 17 & -5 & 13 \\ -8 & 1 & -7 & 1 & -6 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix} \end{aligned}$$

  and

$$\begin{aligned} L^* &= [L_1^*, L_2^*, L_3^*] \\ &= \tfrac{1}{4}\begin{pmatrix} -5 & -2 & 4 & -4 & 1 \\ -7 & -2 & 4 & -8 & 3 \\ -9 & -2 & 0 & -16 & 9 \\ -11 & -2 & -16 & -32 & 27 \\ -13 & -2 & -64 & -64 & 81 \end{pmatrix}, \end{aligned}$$

  yielding

$$RJL = C,$$

  where J (a Jordan matrix) and C are of the form

$$J = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 12 \\ 1 & 0 & 0 & 0 & -40 \\ 0 & 1 & 0 & 0 & 51 \\ 0 & 0 & 1 & 0 & -31 \\ 0 & 0 & 0 & 1 & 9 \end{pmatrix},$$

  which is a Jordan decomposition, as expected.

4 Condition estimation

We shall analyze the sensitivity of the roots of $\pi(t)$ to perturbations in the coefficients a_j viewing the roots as eigenvalues of the associated companion matrix C. Let $\widetilde{\pi}(t)$ denote the perturbed monic polynomial with coefficients $\widetilde{a}_j = a_j + \Delta a_j$ and let $\widetilde{C}$ denote its associated companion matrix. Then, depending on the way the perturbations $\widetilde{a}_j$ are measured, different condition numbers for λ can be obtained. Suppose for instance that the Δa_j's are assumed to satisfy the componentwise inequalities

$$|\Delta a_j| \le \epsilon \alpha_j, \; j = 1, \ldots, m-1, \tag{13}$$

where α_j are arbitrary non negative real numbers. Similarly we denote by $\widetilde{\lambda}_j$, $j = 1, \ldots d$ the eigenvalues of $\widetilde{C}$ corresponding to λ for ϵ small enough. We set $|\Delta\lambda| = \max_{j=1,\ldots,d} |\lambda - \widetilde{\lambda}_j|$. For the so-called *componentwise* model of perturbations defined by (13), we have the definition below, where for simplicity, λ_l and its corresponding multiplicity m_l will be denoted by λ and d, respectively.

Definition 4.1 ([4]) *The* componentwise relative *condition number of the root λ of multiplicity d is defined by*

$$\kappa^{\mathcal{C}}(\lambda) = \lim_{\epsilon \to 0} \sup_{|\Delta a_j| \le \epsilon \alpha_j} \frac{|\Delta\lambda|}{|\lambda|\epsilon^{1/d}}. \tag{14}$$

A precise description of this condition number is the subject of the following proposition.

Proposition 4.2 *Suppose the perturbations Δa_j satisfy (13). Then the componentwise relative condition number of the root λ of multiplicity d, $\kappa^{\mathcal{C}}(\lambda)$, is*

$$\kappa^{\mathcal{C}}(\lambda) = \frac{1}{|\lambda|}\left(\frac{d! \sum_{j=0}^{m-1} |\lambda^j| \alpha_j}{|\pi^{(d)}(\lambda)|}\right)^{1/d} \tag{15}$$

Proof. A sketch of the proof is as follows. Let $\{\widetilde{\lambda}, \widetilde{r}\}$ be a right eigenpair of $\widetilde{C} = C + \Delta C$, with $\widetilde{\lambda} = \lambda + \Delta\lambda$, $\widetilde{r} = r + \Delta r$. Then $(C + \Delta C)(r + \Delta r) = (\lambda + \Delta\lambda)(r + \Delta r)$ iff $C\Delta r + \Delta C r + \Delta C \Delta r = \lambda \Delta r + \Delta\lambda r + \Delta\lambda\Delta r$. Using the fact that the right eigenvector $\widetilde{r}$ of $\widetilde{C}$ is (see Prop. 2.2)$r + \Delta r = \widetilde{H}\phi(\widetilde{\lambda})$, where $\widetilde{H} = H + \Delta H$ has the same structure as H but with entries $\widetilde{a}_j = a_j + \Delta a_j$, it can be proved that the m-th component of Δr equals zero. Next, use this observation and the fact that $\Delta C = -\Delta a e_m^{[m]*}$ where $\Delta a = [\Delta a_0, \; \Delta a_1 \ldots, \Delta a_{m-1}]^T$ and $e_m^{[m]}$ is the m-th canonical vector in $\mathbb{R}^m$, to prove that

$$\Delta C \Delta r = 0. \tag{16}$$

Some algebraic manipulations lead then to the following first order result

$$\Delta\lambda^d = \frac{d!\,\phi^T(\lambda)\Delta a}{\pi^{(d)}(\lambda)}, \qquad (17)$$

from which the proof follows. □

Remark 1. Another condition number for λ can be readily obtained if the perturbations are assumed to satisfy

$$\|\Delta a\|_2 \leq \delta\alpha, \qquad (18)$$

where α is an arbitrary positive real number (e.g., $\alpha = \|a\|_2$). This gives rise to the so-called *normwise relative condition number* $\kappa(\lambda)$. It is immediate that

$$\kappa(\lambda) = \frac{1}{|\lambda|}\left(\frac{d!\|\phi(\lambda)\|\|\alpha\|}{|\pi^{(d)}(\lambda)|}\right)^{1/d} \qquad (19)$$

Remark 2. When the perturbations are measured in a normwise absolute sense. i.e., when $\alpha = 1$ in (18), a similar procedure leads to the so-called *normwise absolute condition* $\kappa_a(\lambda)$ which can be shown to be

$$\kappa_a(\lambda) = \left[\frac{d!\|\phi(\lambda)\|_2}{\pi^{(d)}(\lambda)}\right]^{1/d}. \qquad (20)$$

Proposition 4.3 *Assume the same hypothesis as in the previous proposition. Let* $\sec\Theta_\lambda$ *be the secant of the angle between the leading left generalized eigenvector and the last right generalized eigenvector associated with the eigenvalue* λ*, as defined in Prop. 2.2, i.e.,*

$$\sec\Theta_\lambda = \frac{\|\phi(\lambda)\|_2\|r_d\|_2}{|\phi^T(\lambda)r_d|} \qquad (21)$$

and define

$$\omega_k = |\lambda|^2 + 2|\lambda|\cos\left(\frac{m-d+k}{m-d+k+1}\right), \quad k = 1,\ldots,d.$$

Assume also that the perturbation Δa *satisfy the model (18) with* $\alpha = \|a\|_2$*. Then the condition number* $\kappa(\lambda)$ *given in (19) satisfies*

$$\kappa(\lambda) \leq \tfrac{1+|\lambda|}{|\lambda|}\left(\frac{\|a\|_2^2}{1+\|a\|_2^2}\right)^{\frac{1}{2d}}[\sec\Theta_\lambda]^{1/d} \qquad (22)$$

Proof: Omitted because of space limitation.

There exists a close relationship between $\sec\Theta_\lambda$ and the Wilkinson number of λ when regarded as eigenvalue (simple) of the companion matrix C_{d-1} in (11) (or equivalently, as simple root of $\pi_{d-1}(t)$, see the remark after Prop. 3.2). This can be seen as follows. Since λ is an eigenvalue simple of this matrix, the Wilkinson condition number of λ is [12]

$$\kappa_{\mathrm{w}}(\lambda) = \frac{\|\psi(\lambda)\|_2\|\check{r}_d\|_2}{|\psi^*(\lambda)\check{r}_d|}.$$

From this, because $\phi^T(\lambda)r_d = \psi^*(\lambda)\check{r}_d$ (see (11) again), it is clear that

$$\sec\Theta_\lambda = \kappa_{\mathrm{w}}(\lambda)\frac{\|\phi(\lambda)\|_2}{\|\psi(\lambda)\|_2}. \qquad (23)$$

It turns out that if the the multiplicity d is not large and $|\lambda|$ is a moderate number then

$$\sec\Theta_\lambda \approx \kappa_{\mathrm{w}}(\lambda),$$

in which case $\kappa(\lambda)$ essentially depends on $\kappa_{\mathrm{w}}(\lambda)$. Thus if λ is a well conditioned eigenvalue of the deflated companion matrix C_{d-1} (or equivalently, a well conditioned simple root of $\pi_{d-1}(t)$) and the ratio $\|\phi(\lambda)\|_2/\|\psi(\lambda)\|_2$ is rather small, then moderate values for $\kappa(\lambda)$ may be expected. However, even if $\kappa(\lambda)$ is small, the error in λ will be determined by the multiplicity d and the size of Δa_j. In general, if the perturbations Δa_j are small enough, the relative error in λ can be estimated by the rule

$$\frac{|\Delta\lambda|}{|\lambda|} \approx \kappa(\lambda)\delta^{1/d}. \qquad (24)$$

while the absolute error by

$$|\Delta\lambda| \approx \kappa_a(\lambda)\delta^{1/d}. \qquad (25)$$

4.1 Numerical illustration: Condition estimation

We consider the polynomial $\pi(t)$ of degree $m = 20$ defined by

$$\pi(t) = (t-\lambda)^5(1+t+\cdots+t^{15})$$

with $\lambda = (1+9s)+si$, $0 \leq s \leq 2$. This example is designed to illustrate the role of the deflated polynomial π_{d-1} (in this case $d = 5$, see Prop. 3.2) in estimating the sensitivity of a multiple root. In fact, as in in this case the deflated polynomial $\pi_{d-1}(t) = (t-\lambda)(1+t+\cdots+t^{15})$ reduces to the polynomial $t^{16}-1$ when $\lambda = 1$, all roots of which are known to be extremely well-conditioned [7, Example 4.3], small condition condition numbers for the multiple root λ (as root of $\pi(t)$) can be expected provided that $\lambda \approx 1$, the conditioning being more favorable for the (simple) roots of $\pi(t)$ (the roots of $1+t+\cdots+t^{15}$). Indeed, if the simple roots of $\pi(t)$ are denoted by $\check{\lambda}_k$, it is not difficult to prove that

$$\kappa_a(\lambda) = \frac{(1+|\lambda|^2+|\lambda|^4+\cdots+|\lambda|^{38})^{0.1}}{\prod_{k=1}^{15}|\lambda-\check{\lambda}_k|^{0.2}},$$

whereas for simple roots one has

$$\kappa_a(\check{\lambda}_k) = \frac{\sqrt{5}}{8}\frac{|\check{\lambda}_k-1|}{|\check{\lambda}_k-\lambda|^5}.$$

Some numerical results displayed in Tables 1 and 2 corresponding to several λ's confirm the theoretical prediction. The tables include condition numbers, the predicted eigenvalue errors described in

(24) and (25), and the ratio $\rho = \|\phi(\lambda)\|_2/\|\psi(\lambda)\|_2$. Also, and mainly to verify the theoretical prediction of the error, approximate roots obtained from polynomials with coefficients $\widetilde{a}_j = a_j + \Delta a_j$ where Δa_j are random numbers satisfying a normwise relative error $\delta = 10^{-10}$, are displayed in Figure 1. All computations were performed using MATLAB.

λ	$\kappa(\lambda)$	$\kappa_a(\lambda)$	$\kappa_{\rm W}$
$19 + 2\imath$	$1.3169e + 1$	$1.0480e + 1$	$3.7970e + 0$
$15 + 1.5\imath$	$1.0724e + 1$	$8.6469e + 0$	$3.7443e + 0$
$10 + \imath$	$7.4747e + 0$	$6.2102e + 0$	$3.6221e + 0$
$5 + 0.5\imath$	$3.9220e + 0$	$3.4955e + 0$	$3.2743e + 0$
$1.45 + 0.05\imath$	$1.5800e + 0$	$1.1384e + 0$	$1.7266e + 0$
1	$1.2693e + 0$	$7.7495e - 1$	$1.0000e + 0$

Table 1: Condition numbers.

λ	ρ	$\|\Delta\lambda\|/\|\lambda\|$	$\|\Delta\lambda\|$
$19 + 2\imath$	$1.3322e + 5$	$1.3169e - 1$	$2.5159e + 0$
$15 + 1.5\imath$	$5.1642e + 4$	$1.0724e - 1$	$1.6167e + 0$
$10 + \imath$	$1.0201e + 4$	$7.4747e - 2$	$7.5120e - 1$
$5 + 0.5\imath$	$6.3756e + 2$	$3.9220e - 2$	$1.9707e - 1$
$1.45 + 0.05\imath$	$4.4310e + 0$	$1.5800e - 2$	$2.2924e - 2$
1	$1.1180e + 0$	$1.2693e - 2$	$1.2693e - 2$

Table 2: Ratio ρ and predicted errors .

The results confirm that moderate values of $\kappa(\lambda)$ do not necessarily imply small eigenvalue errors when the multiplicity is rather large and that reasonably small errors can be expected when both the Wilkinson condition number $\kappa_{\rm W}(\lambda)$ and the ratio ρ are small. Further, the theoretical prediction of the eigenvalue error according to (25) displayed in columns 3 and 4 of Table 2 is verified to be consistent with the numerical results as displayed in Figure 1. The relative insensitivity of simple roots is also apparent, as predicted.

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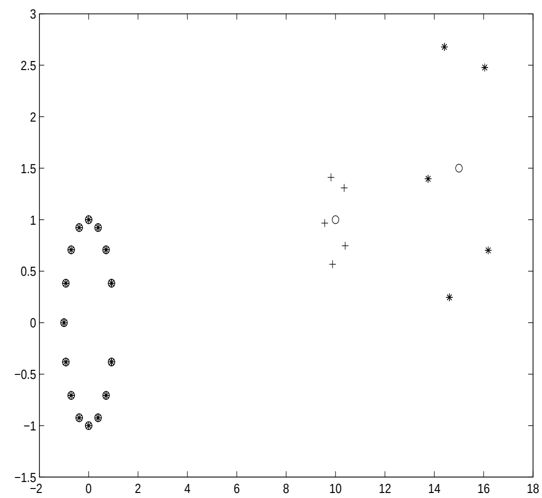

Figure 1: Case 1: $\lambda = 15 + 1.5\imath$. $\circ$: Exact eigenvalue, $*$: Approximate eigenvalue. Case 2: $\lambda = 10 + \imath$. $\circ$: Exact eigenvalue, $+$: Approximate eigenvalue.