Combinatorial and algebraic aspects of Sherman identity

Gustavo A.T.F.da Costa,

Depto de Matemática, UFSC,

88040-900, Florianópolis, SC

E-mail: gatcosta@mtm.ufsc.br

J. Variane Jr.

Instituto de Física Gleb Wataghin - UNICAMP

13083-970-Campinas-SP

1. Introduction

Feynman identity relates admissible graphs and classes of nonperiodic paths on a square lattice. It is an important ingredient in a combinatorial formulation of the Ising model in two dimensions much studied in physics. This identity appeared as a conjecture in lecture notes by R. Feynman, published much later in the book [4]. See [7, 3, 4] for more information on Feynman identity. In [7, 9] Sherman gave it a formal proof. A generalization to any graph appeared recently in [6]. In [8] Sherman considered the identity on a flower like graph with R oriented lines all hooked at one site. In this case the identity can be expressed as:

$$\prod_{m}(1+z_1^{m_1}...z_R^{m_R})^{N_+}(1-z_1^{m_1}...z_R^{m_R})^{N_-}=\prod_{j=1}^{R}(1+z_j) \tag{1.1}$$

where the product on the left is over all integers $m_1,...,m_R \geq 0$. This special case is called Sherman identity in this presentation. A path on this lattice is an ordered sequence of the lines. It has an orientation which is not necessarily that of the lines. A path is nonperiodic if it is not the repetition of some subpath of it two or more times. Paths which are inversions or circular permutations of each other are taken to be equivalent. They form an equivalence class. Call $N_+(m_1,...,m_R)$ and $N_-(m_1,...,m_R)$ the numbers of distinct classes of equivalence of nonperiodic paths with positive and negative signs, respectively, which traverse m_1 times loop 1, m_2 times loop 2, ..., m_R times loop R.

2. The problem

In [8] Sherman raised the problem of interpreting (1.1) in algebraic terms after pointing out its resemblance with the Witt identity of Lie algebra theory. In this identity the exponent have the algebraic meaning of a dimension of a certain vector space associated to some Lie algebra. Interpreting (1.1) in algebraic terms means to find a similar connection for the exponents in the identity. The investigation of this problem is relevant because this may help to understand better the structure of Feynman identity in general and in some new ways. The results of our investigation about Sherman's problem are presented here. It is proved that the identity is actually a consequence of the denominator identity of a Lie algebra.

3. Method of solution

In order to solve the problem one has to compute the numbers $N_\pm$. This can be done as shown in section 3 using a representation of a path by a word which has encoded in it all relevant information about the path. Then one counts the words which represent paths with a positive (negative) sign. Formulas are then derived from this counting. In section 3, these formulas are found to have the structure of a Witt generalized formula which makes the connection with

Lie algebras and help to solve the problem. The results are given in Theorem 1 of section 2, Theorem 2, relation (3.16) and Theorem 3 of section 3.

4. The exponents

On the left hand side of (1.1) there are sequences $(m_1, ..., m_R)$ with $m_i = 1$, $m_{j\neq i} = 0, i = 1, ..., R$ and $N_+ = 1$ and $N_- = 0$ which give the right hand side. To keep formulas as simple as possible let's take all $z's$ equal in (1.1). Then, identity (1.1) can be expressed in the following equivalent way:

$$\prod_{r=2}^{R}\prod_{G_r}\prod_{N=r}^{\infty}(1+z^N)^{\theta_+(N,r)}(1-z^N)^{\theta_-(N,r)} = 1 \tag{4.1}$$

The second product runs over all subdiagrams G_r of G_R with r loops and $\theta_+(N,r)$ and $\theta_-(N,r)$ are the numbers of distinct classes of equivalence of nonperiodic paths with positive and negative signs, respectively, which traverse G_r.

It will be proved that the infinite product in (4.1) converges to 1 on every subgraph G_r, $\forall r \geq 2$. To see that one has to compute the exponents. This goes as follows. First, a path p is represented by a word to be defined below. It gives information about p like the loops it traverses, how many times and in which orientation. From this data it is possible to derive a formula for the sign of p which help one to count the number of nonperiodic words with a positive (negative) sign that gives θ_+ (θ_-).

A word is an ordered sequence of letters $D_i^{e_i}$ where i gives the loop of G_R traversed by p and $\mid e_i \mid= m_i$, how many times. The sign of e_i indicates whether the loop is traversed following the direction assigned for it (in this case, the sign is positive) or the opposite direction (in this case, the sign is negative). A typical word is the following:

$$\mathcal{W}(p) = D_{i_1}^{e_{i_1}} D_{i_2}^{e_{i_2}} ... D_{i_l}^{e_{i_l}} \tag{4.2}$$

where $l = r, r+1, ..., N$. The order in which the letters appear in the word is important since it indicates the loops traversed by p and in which order. The order is encoded in the sequence $S_l = (i_1, i_2, ..., i_l)$. The sequence S_l is such that a loop i appears at least once in the sequence, $i_k \neq i_{k+1}$ and $i_l \neq i_1$. After traversing i_l the path joins i_1. Fix $i_1 = 1$, from now on.

The word has encoded in its data about p information about the sign of p. Next, a formula for the sign is obtained. Given the word (2.2) the sequence $(i_1, i_2, ..., i_l)$ can be decomposed into subsequences defined as follows. A subsequence is an ordered set of numbers formed in such a way that **a)** if i, j are two elements inside the subsequence and j comes after i then $j > i$; **b**) a new subsequence begins whenever this ordering is broken in the sequence by two adjacent elements not satisfying **a)**. For instance, a sequence with $l = 14$ for the case $r = 3$ is decomposed as follows:

$$(12123213232312) = (12)(123)(2)(13)(23)(23)(12) \tag{4.3}$$

Denote by T the number of subsequences in a decomposition. In the example above $T = 7$. Call s the number of negative e_i in $\mathcal{W}(p)$.

Lemma : Given the word $\mathcal{W}(p)$ for p, the sign of p is given by

$$(-1)^{N+l+s+T+1} \tag{4.4}$$

Proof: Following [3], section 2, relation (4.4) is obtained from the representation of p in terms of an appropriate closed normal plane curve compatible with $\mathcal{W}(p)$ which winds the loops of G_R according to some rules. The path is drawn in a extended way, that is, whenever the path should traverse a line more than once the repeated lines are drawn slightly separated so that the path describes a normal closed curve compatible with its word. A normal closed curve on the plane is a curve with no singularities other than a finite number of crossing points where the curve intersects itself at a right angle. By a theorem of Whitney, the sign of p is $(-1)^V$ where V is the number of selfcrossings of its normal curve. Then, just by looking at it the number V is counted. An expression for V in terms of N, s, l and T is obtained which upon substitution in Whitney's formula imply (4.4).

Formulas for the exponents $\theta_\pm$ are given next.

Theorem 1. Given r, $N \geq r$, the number of equivalence classes of nonperiodic paths of length N and positive sign which traverses r loops of G_R is given by

$$\theta_+(N,r) = \sum_{odd\ g|N} \frac{\mu(g)}{g} \mathcal{F}_r\left(\frac{N}{g}\right) \qquad (4.5)$$

where the summation is over the odd divisors of N only and

$$2y\mathcal{F}_r(y) = \sum_{k=-1}^{r-1} (-1)^{r+k+1} \left(\begin{array}{c} r \\ k+1 \end{array} \right) (2k+1)^y \qquad (4.6)$$

and $\mu(g)$ is Möbius function [2]. For the number $\theta_-(N,r)$ of equivalence classes of nonperiodic paths of length N and negative sign which traverses r loops of G_R, the following cases hold:

I) If N is **a)** odd or prime, or **b)** even but $N < 2r$, then

$$\theta_-(N,r) = \theta_+(N,r) \qquad (4.7)$$

II) If N is even and $N \geq 2r$, then

$$\theta_-(N,r) = \theta_+(N,r) - \theta_+(\frac{N}{2},r) \qquad (4.8)$$

Furthemore, if $\mid z \mid < (2r+1)^{-1/2}$ the infinite product in (4.1) converges to 1, for any given r.

Proof: Formula (4.5) is proved in the Appendix using ideas from [3] and result (4.4) above. For the proof of I) and II) see the proof of similar relations for the case $R = 2$ in [3]. To prove convergence of the infinite product call $P_{r,n}$ the partial product in (4.1) with N running from r up to n. Using relations (4.7-8) it's found that for $n \geq 2r$,

$$P_{r,n} = \prod_{j=[\frac{n}{2}]+1}^{n} (1 - z^{2j})^{\theta_+(j,r)} \qquad (4.9)$$

Then, from (4.5) one finds that $lnP_{r,n} \rightarrow 0$ for $\mid z \mid < (2r+1)^{1/2}$ when n goes to infinity, hence $(P_{r,n})$ converges to 1.

5. $\theta_\pm(N,r)$ and Lie algebras

In this section an algebra is found which has the numbers $\theta_+(N)$ as the dimensions of its subspaces and it is then proved that the generalized Witt identity of this algebra implies

$$\prod_{N=r}^{\infty} (1 + z^N)^{\theta_+(N,r)} (1 - z^N)^{\theta_-(N,r)} = 1 \qquad (5.1)$$

The relation

$$M_R(N) = \frac{1}{N} \sum_{g|N} \mu(g) R^{\frac{N}{g}} \qquad (5.2)$$

is well known in Lie algebra theory where it is known as the Witt formula. Witt formula gives the dimensions of the subspaces L_N of a Lie algebra with a $Z_{>0}$-gradation $L = \bigoplus_{N=1}^{\infty} L_N$ generated by an R-dimensional vector space over $\mathbf{C}$. Witt formula satisfies the identity

$$\prod_{N=1}^{\infty} (1 - z^N)^{\dim L_N} = 1 - Rz \qquad (5.3)$$

called the denominator identity for the algebra or simply the Witt identity [5].

Recently [5], Witt formula has been generalized. Let $V = \bigoplus_{i=1}^{\infty} V_i$ be a $\mathbf{Z}_{>0}$-graded vector space over $\mathbf{C}$ with $dimV_i = d(i) < \infty$, $\forall i \geq 1$, and let $L = \bigoplus_{N=1}^{\infty} L_N$ be the free Lie algebra generated by V with a $Z_{>0}$-gradation induced by that of V. Then, the dimensions of the subspaces L_N are given by the generalized Witt formula

$$dimL_N = \sum_{g|N} \frac{\mu(g)}{g} W\left(\frac{N}{g}\right) \qquad (5.4)$$

where W is called the Witt partition function. It is proved in [5] that the numbers $dimL_N$ are such that they satisfy the so called denominator identity or generalized Witt identity

$$\prod_{N=1}^{\infty} (1 - z^N)^{dimL_N} = 1 - f(z) \qquad (5.5)$$

where the function

$$f(z) := \sum_{i=1}^{\infty} d(i) z^i \qquad (5.6)$$

is related with

$$g(z) := \sum_{n=1}^{\infty} W(n) z^n \qquad (5.7)$$

by the relation

$$e^{-g(z)} = 1 - f(z) \qquad (5.8)$$

Given the $d(i)$, W can be computed using a formula from [5]. One then computes $dimL_N$ using (5.4). In the case below W is known and one wants to find a formula for the $d(i)$.

Observe that formula (5.8) for the numbers $\theta_+(N,r)$ looks like (5.4). This sujests that there is a Witt partition function $W(N)$ for these numbers. Indeed, when N is odd,

$$\theta_+(N,r) = \sum_{g|N} \frac{\mu(g)}{g} \mathcal{F}_r\left(\frac{N}{g}\right) \tag{5.9}$$

Thus, for N odd, define

$$W(N) := \mathcal{F}_r(N) \tag{5.10}$$

When N is even, the sum over odd divisors of N in (5.8) can be rewritten using that $\mu(2^n) = 0$, if $n > 2$, and $\mu(2) = -1$ [2] so that

$$\theta_+(N,r) = \sum_{g|N} \frac{\mu(g)}{g} \mathcal{F}_r\left(\frac{N}{g}\right) + \frac{1}{2}\mathcal{F}_r\left(\frac{N}{2}\right) \tag{5.11}$$

Set

$$\theta_+(N,r) = \sum_{g|N} \frac{\mu(g)}{g} W\left(\frac{N}{g}\right) \tag{5.12}$$

Möbius inversion formula and (2.8) imply that

$$W(N) = \mathcal{F}_r(N) + \frac{1}{2}\sum_{g|\frac{N}{2}} \frac{1}{g}\mathcal{F}_r\left(\frac{N}{2g}\right) \tag{5.13}$$

for N even. So Witt partition function has been found and it is given by (5.10) and (5.13) when N is odd and even, respectively. With this W, $\theta_+(N,r)$ can be expressed like (5.4) and for each r the numbers $\theta_+(N,r)$ given by (4.13) can be interpreted as the dimension $dimL_N(r)$ of the subspaces $L_N(r)$ of some algebra $L(r) = \bigoplus_{N=1}^{\infty} L_N(r)$. From the partition function just obtained one can get the $d(i)$'s for any r using a formula given below and from them the function $f(z)$ given by (5.6) and the denominator identity for the algebra.

Theorem 2. The numbers $d(i)$ are given by

$$d(i) = \sum_{m=1}^{i} (-1)^{m+1} \sum \prod_{k=1}^{i} \frac{(W(k))^{a_k}}{a_k!} \tag{5.14}$$

The second summation is over all values of $a_1, ..., a_i > 0$ such that $a_1 + 2a_2 + ... + ia_i = i$ and $a_1 + a_2 + ... + a_i = m$.

Proof: It can be obtained from (5.6) and (5.8) as follows. Formally,

$$d(i) = \frac{1}{i!}\frac{d^i f}{dz^i}(0) = -\frac{1}{i!}\frac{d^i}{dz^i}(e^{-g(z)})|_{z=0} \tag{5.15}$$

Faà di Bruno's relation [1] gives a formula for the i-th derivative of the exponential of a function. From this formula (5.14) follows.

In particular, for $r = 2$,

$$W(1) = 0, W(2) = 2, W(3) = 4, ...$$

and

$$d(1) = 0, d(2) = 2, d(3) = 4, d(4) = 9, ...$$

Call $f_r(z)$ the function (5.6) that corresponds to the Witt partition function given by (5.10) and (5.13). For $r = 2$, $f_2(z) = 2z^2 + 4z^3 + 9z^4 + ...$ The corresponding denominator identity (5.5) is

$$\prod_{N=1}^{\infty} (1 - z^N)^{\theta_+(N,r)} = 1 - f_r(z) \tag{5.16}$$

Set $|z| < (2r+1)^{-1/2}$.

Theorem 3. Identity (5.16) implies (5.1).

Proof: From (5.16) one gets

$$\prod_{N=1}^{\infty} (1 + z^N)^{\theta_+(N,r)} = \frac{1 - f_r(z^2)}{1 - f_r(z)} \tag{5.17}$$

and, using (5.10) and (5.11),

$$\prod_{N=1}^{\infty} (1 - z^N)^{\theta_-(N,r)} = \frac{1 - f_r(z)}{1 - f_r(z^2)} \prod_{N=1}^{r-1} (1 - z^{2N})^{\theta_+(N,r)} \tag{5.18}$$

Then,

$$\prod_{N=1}^{\infty} (1+z^N)^{\theta_+(N,r)}(1-z^N)^{\theta_-(N,r)} = \prod_{N=1}^{r-1} (1-z^{2N})^{\theta_+(N,r)} \tag{5.19}$$

But

$$\prod_{N=1}^{r-1}(1+z^N)^{\theta_+(N,r)}(1-z^N)^{\theta_-(N,r)} = \prod_{N=1}^{r-1}(1-z^{2N})^{\theta_+(N,r)} \quad (5.20)$$

after using that $\theta_-(N,r) = \theta_+(N,r)$ for $N < 2r$. From this (5.1) follows.

Appendix A

Paths of length $N \geq r$ are described by words of the form

$$D_{i_1}^{e_{i_1}} D_{i_2}^{e_{i_2}} ... D_{i_l}^{e_{i_l}} \quad (A.1)$$

where $l = r, r+1, ..., N$, $i_k \neq i_{k+1}, i_k = 1, ..., r$, and

$$\sum_{k=1}^{l} \mid e_{i_k} \mid = N \quad (A.2)$$

The number $S_r(N)$ of such words is given by

$$S_r(N) = \sum_{l=r}^{N} 2^l p_l(N) r w_r(l) \quad (A.3)$$

where

$$p_l(N) = \left(\begin{array}{c} N-1 \\ l-1 \end{array} \right) \quad (A.4)$$

and

$$rw_r(l) = \sum_{j=1}^{r} (-1)^{r+j} \left(\begin{array}{c} r \\ j \end{array} \right) (j-1)^l + (-1)^{l+r} \quad (A.5)$$

$p_l(N)$ is the number of unrestricted partitions of N into l nonzero parts $\mid e_{i_k} \mid$, $k = 1, 2, ..., l$. Since each e_i is either positive or negative there are 2^l ways of assigning $+$ and $-$ signs to these numbers. Call $\alpha_1, \alpha_2, ..., \alpha_r$, the loops of G_r. Given $i_1 \in \{\alpha_1, \alpha_2, ..., \alpha_r\}$, $w_r(l)$ is the number of sequences $(i_1, i_2, ..., i_l)$ with i_1 fixed and $i_k \in \{\alpha_1, \alpha_2, ..., \alpha_r\}$, such that: **a)** each α_k shows up at least once in the sequence; **b)** $i_k \neq i_{k+1}$; **c)** $i_l \neq i_1$. Since there are r possibilities for i_1 we multiply $w_l(N)$ by r to get all possible sequences.

One can extend $w_r(l)$ to allow $l < r$. In this case, $w_r(l) = 0$ and the sumation in (A.3) can start from $l = 1$.

Following the ideas of [3] denote by $\overline{S_r(l/g, N/g, s/g, T/g)}$ the number of nonperiodic words plus their circular permutations associated to the numbers l/g, N/g, s/g, T/g. Then,

$$S_r(l, N, s, T) = \sum_{g|(l,N,s,T)} \overline{S_r(\frac{l}{g}, \frac{N}{g}, \frac{s}{g}, \frac{T}{g})} \quad (A.6)$$

where the summation is over the common divisors g of l, N, s and T. The term in (A.6) with $g = 1$ counts precisely the number of distinct nonperiodic words whereas the $g \neq 1$ terms count the periodic ones.

Applying Möbius inversion formula [2] it follows that

$$\overline{S_r(l, N, s, T)} = \sum_{g|(l,N,s,T)} \mu(g) S_r(\frac{l}{g}, \frac{N}{g}, \frac{s}{g}, \frac{T}{g}) \quad (A.7)$$

where μ is the Möbius function [2]. A formula for $S_r(l, N, s, T)$ can be computed after the the following decompositions in (A.3) are made:

$$2^l = \sum_{s=0}^{l} \left(\begin{array}{c} l \\ s \end{array} \right) \quad (A.8)$$

and

$$w_r(l) = \sum_{T} y_r(T, l) \quad (A.9)$$

where $y_r(T, l)$ is the number of sequences with the same number T of subsequences. Explicit knowledge of y_r is not necessary. From (A.3) it follows that

$$S_r(l, N, s, T) = \left(\begin{array}{c} l \\ s \end{array} \right) \left(\begin{array}{c} N-1 \\ l-1 \end{array} \right) r y_r(T, l) \quad (A.10)$$

Using the result (2.4) from section 2,

$$\theta_{\pm}(N, r) = \sum_{l,s,T} \frac{\overline{S_r(l, N, s, T)}}{l} \quad (A.11)$$

where the sum runs over all l, s and T satisfying the condition that $l + N + s + T$ is odd for θ_+ and $l + N + s + T$ is even for θ_-. In the case of θ_+ one gets after performing the summations

$$\theta_+(N, r) = \sum_{odd\ g|N} \frac{\mu(g)}{g} \sum_{l=1}^{\frac{N}{g}} \frac{1}{l} \left(\begin{array}{c} \frac{N}{g} - 1 \\ l-1 \end{array} \right) 2^{l-1} r w_r(l) \quad (A.12)$$

Using (A.5) and performing the summation over l yields result (4.5-6).

References

[1] M. Abromowitz, I. A. Stegun, Handbook of Mathematical Functions, Dover Publications, New York, 1972.

[2] T. M. Apostol, Introduction to Analytic Number Theory, Springer Verlag, 1986.

[3] G. A. T. F. da Costa, Feynman identiy: a special case. I, J. Math. Phys. **38** (1997), 1014-1034.

[4] R. P. Feynman, Statistical Mechanics. A Set of Lectures, Benjamim/Cummings, New York, 1972.

[5] S. Kang, M. Kim, Free Lie Algebras, Generalized Witt Formula, and the Denominator Identity, J. Algebra **183**(1996), 560-594.

[6] M. Loebl, A discrete non-pfaffian approach to the Ising problem, in "Graphs, morphisms and statistical physics", 145-154, DIMACS Ser. Discrete Math. Theoret. Comput. Sci.,**63**, Amer. Math. Soc., Providence, RI, 2004.

[7] S. Sherman, Combinatorial aspects of the Ising model for ferromagnetism. I. A conjecture of Feynman on paths and graphs, J. Math. Phys. **1** (1960), 202-217.

[8] S. Sherman, Combinatorial aspects of the Ising model for ferromagnetism. II. An analogue to the Witt identity, Bull. Am. Math. Soc. **68** (1962), 225-229.

[9] S. Sherman, Addendum: Combinatorial aspects of the Ising model for ferromagnetism. I. A conjecture of Feynman on paths and graphs, J. Math. Phys. **4** (1963), 1213-1214.