Approximation Problems Categories *

Liara A. dos S. Leal, **Dalcidio M. Claudio**
Faculdade de Matemática - PUCRS,
90619-900, Porto Alegre, RS
E-mail: liara@pucrs.br, dalcidio@pucrs.br

The notion of approximation problems was formally introduced by Johnson [3] in his pioneering paper on the approximation of combinatorial optimization problems, and it was also suggested a possible classification of optimization problems on grounds of their approximability properties. Since then, it was clear that, even though the decision versions of most NP-hard optimization problems are polynomial-time reducible to each other, they do not share the same approximability properties. In spite of some remarkable attempts, according to Ausiello [1] the reasons that a problem is approximable or nonapproximable are still unknown. The different behaviour of NP-hard optimization problems with respect to their approximability properties is captured by means of the definition of approximation classes and, under the "P $\neq$ NP" conjecture, these classes form a strict hierarchy whose levels correspond to different degrees of approximation.

In this paper we continue along the same line of research started in [4], towards to providing a categorical view of structural complexity to optimization problems. The main aim is to provide a universal language for supporting formalisms to specify the hierarchy approximation system for an abstract NP-hard optimization problem, in a general sense. From the observation that, intuitively, there are many connections among categorical concepts and structural complexity notions, we started defining two categories: the OPTS category of polynomial time soluble optimization problems, which morphisms are reductions, and the OPT category of optimization problems, having approximation-preserving reductions as morphisms. The study of approximation implies to create means of comparing optimization problems. The basic idea of approximation by models is a recurrent one in mathematics and in this direction a comparison mechanism between the OPTS and OPT categories has been introduced in [5]. In order to establish a formal ground for the study of the approximation properties of optimization problems, a system approximation to each optimization problem is constructed, based on categorical shape theory. In so doing, we were very much inspired in previous works by Rattray [8, 9] on complex systems.

Given a functor K: OPTS $\longrightarrow$ OPT, the category $APX_{B,K}$ of approximations to an optimization problem $B \in$ OPT is the comma category $B \downarrow K$ of K-objects under B. A such kind of limit construction provides a means of forming complex objects from patterns (diagrams) of simpler objects. In particular, by using co-limits in the $APX_{B,K}$ definition, a hierarchical structure can be imposed upon the system of approximation, reaching the best approximation from the system, if it exists. Besides, optimization problems B and B' can be compared by their approximation $B \downarrow K$ and $B' \downarrow K$ more easily. In addition, if K has an adjoint then each $B \downarrow K$ has an initial object, i.e., a best approximation to B. The advantage of initiality conditions is that they imply that each $B \downarrow K$ can be handled as if it were a directed set. Thus the existence of an initial object means that given any two approximation, one can find a mutual refinement of them.

In a sequel of this paper, we have planned to extend the investigation in order to characterize optimization problems in terms of their hardness in being approximated, also exploiting farther on the class of NPO problems.

After defining both the OPTS and OPT categories in [6], the next step was to identify the relationships between them. In [7] were proposed two basic questions: What does it mean to say that a problem A "approximates" an optimization problem B? What is it understood by the "best approximation" for such an optimization problem?
In order to answer those questions, were provided mechanisms for the comparison between such categories. This led us to the categorical shape theory. The first categorical approach to shape theory arose in the beginning seventies. Since then many other works related to the subject have appeared. As was pointed by Cordier and Porter in their introduction to [2], shape theory describes a process which is common in mathematical reasoning. Typically one has a class of objects in which one has a reasonably complete set of information. This class is considered as a class of "models" or "prototypes" within a larger class of objects of interest.

In the context of categorical shape theory, there are three basic defining elements:

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1. a category **B** of objects of interest;
2. a category **A** of prototypes or model-objects;
3. a "comparison" of objects with model-objects, ie. a functor $K : \mathbf{A} \longrightarrow \mathbf{B}$.

Given category **A** of prototypes, category **B** of objects of interest, and a comparison $K : \mathbf{A} \longrightarrow \mathbf{B}$, an *approximation* to an object B in **B** is the pair (f, A), where A in **A** is an prototype and $f : B \longrightarrow KA$.

A morphism between approximations $h : (f, A) \longrightarrow (g, A')$ is a morphism $h : A \longrightarrow A'$ of the underlying prototypes, such that $K(h) \circ f = g$, ie. the triangle in the figure 1 commutes.

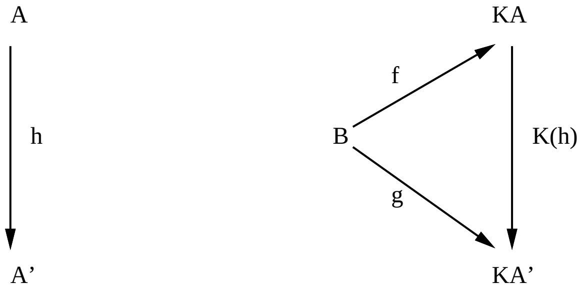

Figura 1: Morphisms between Approximations

If B is an object of **B**, one can form the comma category $B \downarrow K$ whose objects are pairs (f, A) with $f : B \longrightarrow KA$. A morphism from (f, A) to (g, A') is a morphism $a : A \longrightarrow A'$ such that $K(a) \circ f = g$.

If $h : B \longrightarrow B'$ is a morphism in **B**, there is an induced functor $h^* : B \downarrow K \longrightarrow B' \downarrow K$ obtained by composition in an obvious way. This functor preserves the codomain $h^*(f, A) = (fh, A)$.

A *shape category* is defined introducing new morphisms preserving codomain between objects in **B**. The basic idea behind categorical shape theory is that recognizing and understanding an object of interest B via a comparison $K : \mathbf{A} \longrightarrow \mathbf{B}$ requires the identification of the corresponding prototype A which best represents B. Besides, in any approximating situation, the approximations are what encode the only information that it can analyze.
Through categorical shape theory and under a few many conditions it is possible to identify the best approximation to an optimization problem B in OPT category, if it exists. The notion of "most closely approximates" is given by a *universal* object. Besides, it is provided the way of comparing NP-hard problems whose are approximated by the same design technique.

Let OPT be the category of objects of interest **B** and OPTS the category of prototypes **A**. We must have some way of comparing hard problems with tractable problems. The fundamental techniques for the design of approximation algorithms are presented in [1]. In many cases it is possible to define an algorithm scheme that can be applied to obtain several algorithms for the same problem with possibly different approximation properties. The most used such design techniques are: relaxation method, greed method, local search, linear programming based algorithm, dynamic programming and randomized algorithm. Let K: OPTS$\longrightarrow$ OPT be a comparison mechanism related to an approximation method (for instance by using relaxation method). In order to characterize approximation degrees by means of categorical shape theory, the basic idea is the construction of a system approximation to each optimization problem, using the notion of co-limit. In a general case, approximations with their morphisms form a category $B \downarrow K$, the comma category of K-objects under B.

Given a functor K:OPTS$\longrightarrow$OPT, a problem $B \in$OPT is said an *approximation problem* if there are a problem $A \in$OPTS and an approximation-preserving reduction f, such that $f : B \longrightarrow KA$. In this case, the pair (f, A) is an *approximation* to the problem B. Notice that as a particular K may apply distinct problems from OPTS to the same problem in OPT, it is better to represent such an approximation as a pair (f, A).

Given a functor K:OPTS$\longrightarrow$OPT, the category $APX_{B,K}$ of approximations to an optimization problem $B \in$ OPT is the comma category $B \downarrow K$ of K-objects under B.

The cone-like form of the morphisms in **B** giving the approximations for some object B, suggests that taking the limit object of the diagram would result in an prototype A* "as near as possible " to B. See figure 2 below.

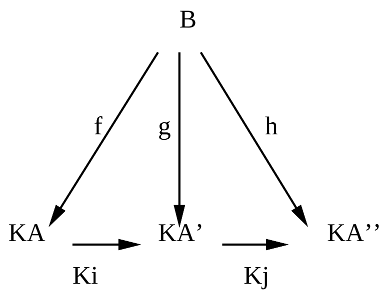

Figura 2: Approximations to B

The definition of a morphism $h : (f, A) \longrightarrow (g, A')$ between approximations corresponds to saying that $g : B \longrightarrow KA'$ can be written as a composite $K(a)^\circ f$, where $f : B \longrightarrow KA$ and $a : A \longrightarrow A'$, that is

$$B \rightarrow KA \rightarrow KA'$$

This it means that (f, A) already contains the information encoded in (g, A'). Thus in some way (f, A) is "finer"approximation to B than is (g, A'). The cone-like form of the morphisms in **B** giving the approximations for some problem B, suggests that the best approximation to such problem B, if it exists, is given by a limit object in $APX_{B,K}$. In this case, a hierarchical structure can be imposed upon the system of approximation by using a kind of universal construction in the category of approximations.

Very often we are faced to comparing two problems related to the approximation issue. Supposing that it is given a comparison functor K:OPTS$\longrightarrow$OPT, and a problem-object B in OPT, the category of approximations $APX_{B,K}$ encodes the only information available on B, by using an approximating-object (f, A). Therefore, if we would compare two problems B and B' in OPT, we should compare the corresponding categories of approximations $APX_{B,K}$ and $APX_{B',K}$.

In this case a morphism preserving codomain from B to B' induces a functor(shape morphism) that compares the information encoded in their corresponding categories of approximations. The meaning of this categorial construction has to be investigated in more detail and it is in order for further work.

In the context of complexity theory, the existence of an algorithm scheme to a problem means that there is a best approximation to such problem.

Consider the comparison functor K:OPTS$\longrightarrow$OPT resulting of an algorithm scheme. In the categorical approach, this it means that a problem-object B in OPT has a K-universal prototype A in OPTS. Therefore there is an adjoint functor to K. This fact implies that the corresponding category of approximations $APX_{B,K}$ has an initial subcategory consisting of a single morphism and a single object. Thus such a category can be handled as if it were a directed set. The meaning of this result is of great theoretical significance: it implies that given any two approximations to the problem B, one can find a finer approximation than both of them.

In conclusion, we note that approximation of optimization problems has become a very active area of research. Nowadays it is known that the computational efficiency of approximating different NP-hard optimization problems varies a great deal. It is normal in computational theory to regard a problem as “tractable” if we know of an algorithm that takes time that is bounded above by some polynomial of the size of the problem instance. Unfortunately, for many problems there are complexity theoretic evidence to suggest strongly that they are, in fact, “intractable”. In order to define the structure of problems better, much effort has been turned to classifying computational problems according to how hard they are to solve. However, computational problems are not only things that have to be solved. They are also objects that can be worth studying problems and can be formalized mathematically. In this paper we extend our previous work on the application of categorical shape theory in order to provide a mathematical framework in dealing with the question outlined above. Our knowledge is by no means complete however, and there remain many open problems. The direction is aimed towards actually exploring the connections among the structural complexity aspects and categorical concepts, which may be viewed in a ”high-level”, in the sense of a structural complexity approach.

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