Galerkin Least Squares approximations for flows of viscoplastic fluids obeying the Casson-Papanastasiou constitutive equation

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Preliminaries

The analysis of non-Newtonian flows has always been a challenge for fluid mechanists. Prediction of fluid behavior and detailed flow visualization in complex geometry, mostly not accomplishable in experimental analysis, has stimulated the research on non-Newtonian computational fluid dynamics.

Our goal is to develop approximations for isochoric flows of viscoplastic fluids. The basic problem is to approximate the motion equations coupled to a rheological model for the extra-stress tensor. Generalized Newtonian models adjust the commonly observed phenomena of pseudoplasticity and shear-thickening, by a shear-rate-dependent equation for the viscosity. Viscoplastic models are those that also fit a yield stress, i.e., a level of shear stress under which the material would not deform, and beyond which the material would have a fluid-like behavior. In many industrial applications of engineering interest, pseudoplasticity and viscoplasticity represent the most relevant non-Newtonian features for predicting flow dynamics Examples of viscoplastic materials are molten chocolate, ketchup, blood, molten polymers and polymeric solutions [1]. The Casson constitutive equation is built to fit rheological data of viscoplastic fluids and is commonly employed in modeling such fluids.

In this work, we approximate the equations of isochoric motion of viscoplastic liquids via a finite element stabilized method, named Galerkin Least Squares (GLS). The need for a stabilized method arises because, in the context of isochoric flows, the numerical approach via the classical Galerkin Method suffers from two major difficulties. First, the need to satisfy Babuška-Brezzi condition with respect to the combinations of functional sub-spaces. Second, the inherent instability of central difference schemes for advective dominated equations. The GLS method provides stability to the original Galerkin formulation by adding terms obtained by minimizing the square of the equation residual. This method has the ability to circumvent Babuška-Brezzi condition and to generate stable approximations for highly advective flows preserving good accuracy properties, retaining the weighted residual structure of the Galerkin formulation and not damaging its consistency [2].

Finite Element Approximation

Assuming the generalized Newtonian fluid with viscosity function $\eta(\dot\gamma)$ [3] one may formulate the Principle of Power Extended [4] to this class of liquids. Based upon the usual fluid dynamic spaces for the velocity and pressure fields [5], a GLS formulation for isochoric flows runs as follows: *Find the pair* $(\mathbf{u},p)\in \mathbf{V}_g^h \times P^h$, $\forall\ (\mathbf{v}, q) \in \mathbf{V}^h \times P^h$, *such as:*

$$\begin{aligned}
&\int_\Omega [\operatorname{grad}\mathbf{u}]\mathbf{u}\cdot \mathbf{v}\,d\Omega + \int_\Omega 2\,\mathrm{Re}^{-1}\,\eta^*(\dot\gamma^*)\mathbf{D}(\mathbf{u})\cdot\mathbf{D}(\mathbf{v})\,d\Omega + \\
&-\int_\Omega p\operatorname{div}\mathbf{v}\,d\Omega - \int_\Omega q\operatorname{div}\mathbf{u}\,d\Omega \\
&+\sum_{\Omega_K\in C_h}\int_{\Omega_K}\Big([\operatorname{grad}\mathbf{u}]\mathbf{u} + \operatorname{grad}p - 2\,\mathrm{Re}^{-1}\operatorname{div}\big(\eta^*(\dot\gamma^*)\mathbf{D}(\mathbf{u})\big)\Big)\cdot \\
&\Big(\tau(\mathrm{Re}_K)([\operatorname{grad}\mathbf{v}]\mathbf{u} - 2\,\mathrm{Re}^{-1}\operatorname{div}\big(\eta^*(\dot\gamma^*)\mathbf{D}(\mathbf{v})\big) - \operatorname{grad}q)\Big)d\Omega_K = \\
&=\int_\Omega \mathrm{Fr}^{-2}\mathbf{f}\cdot\mathbf{v}\,d\Omega + \int_\Gamma \mathbf{t}\cdot\mathbf{v}\,d\Gamma + \sum_{\Omega_K\in C_h}\int_{\Omega_K}\mathrm{Fr}^{-2}\mathbf{f}\cdot \\
&\Big(\tau(\mathrm{Re}_K)\Big([\operatorname{grad}\mathbf{v}]\mathbf{u} - \operatorname{grad}q - 2\,\mathrm{Re}^{-1}\operatorname{div}\cdot\big(\eta^*(\dot\gamma^*)\mathbf{D}(\mathbf{v})\big)\Big)\Big)d\Omega_K
\end{aligned}$$

where $\mathbf{u}$ and p are the finite element approximations for velocity and pressure, respectively, Re the Reynolds number, $\mathbf{D}$ the strain rate tensor, q the pressure variation, $\mathbf{v}$ the virtual velocity, Fr the Froude number, $\mathbf{f}$ the body force and $\mathbf{t}$ contact force. The τ-parameter in the least-square terms of equation above is the same as in Franca & Frey [2] for linear Newtonian flows.

Numerical Results

In this section, the GLS approximation of the steady-state flow of a viscoplastic fluid through planar 4:1 expansion will be presented (see Fig. **1** for the problem statement). We use the Casson equation [6] to model the extra-stress tensor for a viscoplastic material, employing Papanastasiou´s approximation [7] to overcome the difficulties of the discontinuity of

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the original model. Such model, called herein Casson-Papanastasiou, is given by:

$$\eta^{1/2} = \eta_0^{1/2} + (\tau_0/\dot{\gamma})^{1/2}\left[1-\exp\left(-(m\dot{\gamma})^{1/2}\right)\right]$$

where τ_0 is the yield stress and η_0 the plastic viscosity. This equation is valid both in yielded and unyielded regions, and the exponent m controls the smoothness of the function. For $m \geq 100$ it mimics the original Casson model. The dimensionless Casson and Reynolds numbers are:

$$\mathrm{Ca} = \frac{\tau_0 L}{\eta_0 u_0}; \quad \mathrm{Re} = \frac{L u_0 \rho}{\eta_0}$$

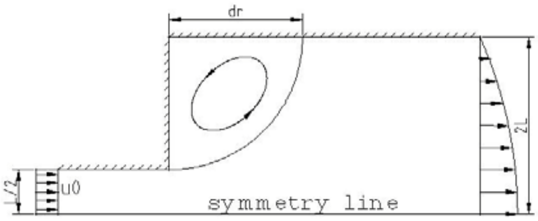

Figure **1** Planar 4:1 expansion

All simulations were performed using a finite element code developed by the research group of the Laboratory of Applied and Computational Fluid Mechanics of the Mechanical Engineering Department of UFRGS. The results shown herein are those obtained for a mesh of 6200 Q_1/Q_1 finite elements. A Newton-like algorithm was applied for the solution of the resulting non-linear algebraic system. For the investigated flows, the Reynolds number was equal to 1, 10 and 50, and the Casson number was varied between 0 (Newtonian fluid) and 10.

Figure **2** shows the growth of the unyielded zone for increasing Ca number, for Re=10. Similar behavior was observed for the other values of Re investigated.

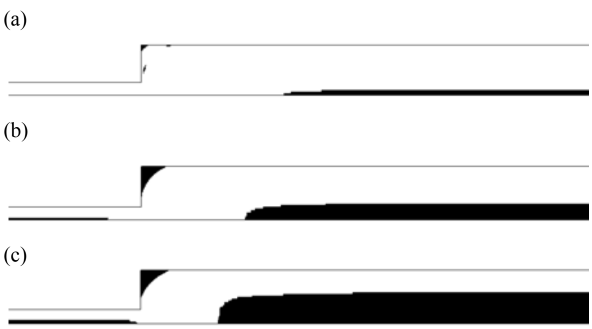

Figure **2**: Unyielded zones (in black), Re=10. (a) Ca=0.1, (b) Ca=1, (c) Ca=10

The unyielded region right downstream the expansion may be a drawback for the formation of a vortex. When the Casson number is sufficiently small, a vortex still appears. That behavior was analyzed by the calculation of streamlines. It is possible to note that for the fluid to form a vortex, it is necessary that a sufficient high Re is combined with a sufficient low Ca. In the cases of low Re, i.e., Re=1 and Re=10, the recirculation is mild even for a Newtonian fluid, and viscoplasticity causes the flow to have a stagnant character right downstream the expansion. The competition between inertia and viscoplasticity is clear: as Ca is increased, the unyielded region is enhanced and the vortex diminishes as it is shown in Fig. 3.

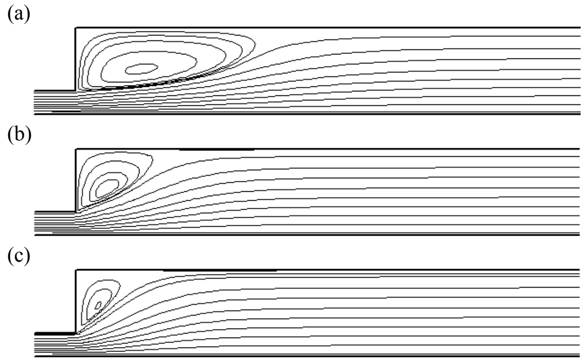

Figure **3**: Streamlines, Re=50. (a) Ca=0.1, (b) Ca=0.8, (c) Ca=1.5

In Fig. 3, it is possible to observe that, in contrast with the Newtonian model, the vortex grows from the corner of the expansion but from the exit border of the smaller tube. The vortex enhancement with increasing Re, behavior also observed for Newtonian fluids, is compensated by the rigid zone enhancement which opposes vortex growth. The topology of the rigid zone may be a good explanation for the distinct shape of the recirculation when comparing a viscoplastic and Newtonian flows, as depicted in Fig. 4.

Figure **4**: Structure of the unyielded zone, Re=50. (a) Ca=0.1, (b) Ca=0.8, (c) Ca=1.5

One may notice that the unyielded zone is cut in two parts, one in the corner and other at the stagnation point between the vortex and the main flow, which grows until a unique unyielded rigid zone is formed for the highest Ca. This phenomena was first observed by Jay et al. [8] in their numerical investigation using a Hershell-Bulkley model in axisymmetric flow. These authors confirmed the phenomena observing an equal behavior in a set of experimental analysis employing a viscoplastic material. It is interesting to visualize the same kind of phenomena in the case of planar flow.

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