Abstract
In this work, bounded lattices are considered from the point of view of interval fuzzy logic. Particularly, the article focuses mainly on the notions of t-conorms, fuzzy negations and S-implication from the unit interval to an arbitrary bounded lattice. Their interval extensions motivated us to investigate some general properties of S-implications on bounded lattices. The interval extensions of t-conorms, fuzzy negations and S-implications on bounded lattices preserve the optimality property and reports to the best interval representations of these fuzzy connectives.

1 Introduction
Fuzzy set theory [29] may be thought as having arisen from the need of a more complete and inclusive mathematical model of uncertainty. Interval analysis [22] arose out of a need to understand and model the uncertainty and the error in numerical computations, allowing the development of computational tools for the automatic error analysis of numerical algorithms running in digital computers.

Based on this foundations, the use of lattice theory to deal with fuzzy logic in a more general framework has been increased. For instance, several works including L-fuzzy set theory [16], BL-algebras of Hájek [17] and Brouwerian lattices [26] may be found. A t-norm notion bounded to partially ordered sets, which are more general structure than the bounded lattices, is considered in [11, 10]. In [24, 7], an extension of t-norms for bounded lattice was presented in the same sense as proposed by [11, 10].

This paper is laid in the intersection of three areas, namely, interval mathematics, fuzzy logic and bounded lattices. Different from the papers mentioned in the above paragraph, we are taking into account the usual interval constructor on bounded lattice to obtain a formal characterization for t-conorms, fuzzy negations and fuzzy implications on arbitrary bounded lattices.

The aim of this work is to introduce a generalization of t-conorms, fuzzy negation and fuzzy S-implications for arbitrary bounded lattices. It is shown that the interval constructor preserves the usual constructions of S-implication as well as some properties satisfied by this connective.

The paper is organized as follows. Section 2 introduces a brief discussion about the benefits obtained with the interval fuzzy logic and related works. In Sect. 3 we consider the well known notion of bounded lattice and the interval constructor on this class of lattices. In Sect. 4, we dis-
cuss the conditions to obtain the best interval representation of a real function and present the related definitions and results. Based on these considerations, we focus attention on the interval extensions of fuzzy t-conorm and fuzzy negation in Sections 5 and 6, respectively. Further analysis of the properties satisfied by fuzzy S-implications is done in Sect. 7. Section 7.1 shows that minimal properties of fuzzy implications may be extended from interval fuzzy degrees, in a natural way. In addition, in Sect. 7.2, a commutative diagram relating fuzzy S-implications with interval fuzzy S-implications is also discussed. In Sect. 8, we conclude with the main results of this paper and final remarks.

2 Interval Fuzzy Logic

Fuzzy systems technology has achieved its maturity with widespread applications in many industrial, commercial, and technical fields, ranging from control, automation, and artificial intelligence to image/signal processing, pattern recognition, and electronic commerce.

On the other hand, the correctness and optimality of interval mathematics have been applied in technological [3] and scientific computations [12] to provide accuracy of calculations together with automatic and rigorous control of errors in numerical computations [18].

As another form of information theory which is related to but independent from fuzzy logic, interval mathematics has reached its current outstanding status, providing reliable techniques to deal with the imprecision of the input parameters and data, and to control the round errors that occur and propagate during the computation [22, 1].

There are several papers connecting these areas (see, e.g., [23, 9, 21, 13, 15]), showing that the intrinsic relation between fuzzy logic and interval mathematics is certainly of significant interest and of potential importance.

This paper is based on Bedregal and Takahashi work’s [5, 6], where interval extensions for the fuzzy connectives, considering both correctness (accuracy) and optimality aspects, were provided [25]. In this approach, the specialists’ uncertainty in the determination of a certain membership degree is represented by an interval membership degree.

The extension of classical logic connectives to the real unit interval is fundamental for the studies on fuzzy logic and therefore is essential to the development of fuzzy systems. This extension must preserve the behaviors of the connectives at the interval endpoints (crisp values) and important properties, such as commutative and associative properties, which result in the notions of triangular norms and triangular conorms.

Fuzzy implications play an important role in fuzzy logic, both in the broad sense (heavily applied to fuzzy control, analysis of vagueness in natural language and techniques of soft-computing) and in the narrow sense (developed as a branch of many-valued logic which is able to investigate deep logical questions). However, there is no consensus among researchers which extra properties fuzzy implications should satisfy. In the literature, several fuzzy implication properties have already been considered and their interrelationship with the other kinds of connectives are generally presented. There exist three main classes of fuzzy implications associated to fuzzy connectives, namely, R-implications, S-implications and QL-implications, which are generated by t-norms, t-conorms joint with fuzzy negations and t-norms together with t-conorms and strong fuzzy negations, respectively.

3 Lattices

Let \( L = (L, \land, \lor) \) be an algebraic structure, where \( L \) is a nonempty set and \( \land \) and \( \lor \) are binary operations, called, respectively, meet and joint. \( L \) is a lattice, if, for each \( x, y, z \in L \), the following conditions hold:

1. \((a)\) \( x \land y = y \land x \) and 
   \((b)\) \( x \lor y = y \lor x \);

2. \((a)\) \( x \land (y \land z) = (x \land y) \land z \) and 
   \((b)\) \( x \lor (y \lor z) = (x \lor y) \lor z \);

3. \((a)\) \( x \land (x \lor y) = x \) and \((b)\) \( x \lor (x \land y) = x \).

Thus, a lattice \( L = (L, \land, \lor) \) is defined as an algebra on a nonempty set with binary operations join and meet, which are commutative and associative, and satisfy the absorption laws.

If there exist two distinct elements, 0 and 1, such that, for each \( x \in L \), \( x \land 1 = x \) and \( x \lor 0 = x \), then \( (L, \land, \lor, 1, 0) \) is said to be a bounded lattice.

In addition, each lattice establishes a partial order. Let \( L = (L, \land, \lor) \) be a lattice. Thus, the binary relation \( \leq_L \subseteq L \times L \) defined by

\[
x \leq_L y \iff x \land y = x \iff x \lor y = y
\]
is a partial order, where \( \land \) is the greatest lower bound (infimum) and \( \lor \) with the least upper bound (supremum) of the set \( \{x, y\} \).

Let \( L \) be a bounded lattice. In the interval version of \( L \), denoted by \( IL = \{IL, \land, \lor, [1, 1], [0, 0]\} \), the set of intervals in \( L \), \( IL = \{[x, \bar{x}] : x, \bar{x} \in L, \bar{\bar{x}} \leq \bar{x}\} \), satisfies the conditions

\[
[x, \bar{x}] \land [y, \bar{y}] = [x \land \bar{y}, \bar{x} \land \bar{y}]
\]

and

\[
[x, \bar{x}] \lor [y, \bar{y}] = [x \lor \bar{y}, \bar{x} \lor \bar{y}].
\]

Thus, \( IL \) is also a bounded lattice.

A characterization of the associated order for a bounded lattice that agrees with the product order can be expressed as

\[
[x, \bar{x}] \leq [y, \bar{y}] \text{ iff } x \leq_L y \text{ and } \bar{x} \leq_L \bar{y}.
\]  

(1)

The partial order in Eq. (1) generalizes the partial order introduced in the work of Kulisch and Miranker [20], in the context of interval mathematics.

4 Interval constructor

The notions of interval representation and canonical interval representation, here refereed as the best interval representation, were both introduced in [25], with the goal of providing a mathematical foundation for interval computations, where correctness and optimality principles can be verified. In this section, a generalization for arbitrary bounded lattices is presented.

Notice that an interval \([x, \bar{x}] \in IL\) may be seen as a pair of elements \((x, \bar{x})\) or as a set of elements \(\{x \in L : x \leq_L x \leq_L \bar{x}\}\).

An interval \(X \in IL\) is said to be an interval representation of each \(\alpha \in X\). Let \(X\) and \(Y\) be interval representations of \(\alpha\). Thus, \(X\) is said to be a better representation of \(\alpha\) than \(Y\) if \(X \subseteq Y\). This notion can be easily extended for \(n\)-uples of intervals, indicated by \((\vec{X}) = (X_1, \ldots, X_n)\).

Definition 4.1 A function \(F : IL^n \rightarrow IL\) is an interval representation of a function \(f : L^n \rightarrow L\) if, for each \(\vec{X} \in IL^n\) and \(\vec{x} \in \vec{X}\), \(f(\vec{x}) \in F(\vec{X})\).

Thus, an interval function \(F : IL^n \rightarrow IL\) is a better interval representation of the function \(f : L^n \rightarrow L\) than \(G : IL^n \rightarrow IL\), denoted by \(G \subseteq F\), if for each \(\vec{X} \in IL^n\), the inclusion \(F(\vec{X}) \subseteq G(\vec{X})\) holds.

4.1 Best interval representation

Definition 4.2 For each function \(f : L^n \rightarrow L\), the interval function \(\hat{f} : IL^n \rightarrow IL\), defined by

\[
\hat{f}(\vec{X}) = [\inf \{f(\vec{x}) : \vec{x} \in \vec{X}\}, \sup \{f(\vec{x}) : \vec{x} \in \vec{X}\}]
\]

(2)

is called the best interval representation of \(f\).

The interval constructor \(\hat{f}\) is well defined and for any other interval representation \(F\) of \(f\), \(F \subseteq \hat{f}\). The interval function \(\hat{f}\) returns a narrower interval than any other interval representation of \(f\). Thus, \(\hat{f}\) has the optimality property of interval algorithms mentioned by Hickey et al. [18], when it is seen as an algorithm to compute a real function \(f\).

5 T-conorm on bounded lattice and the interval constructor

It follows from the work proposed in [5] that an interval triangular conorm (t-conorm, for short) may be considered as an interval representation of a t-conorm. This interval generalization of a t-conorm on the unit interval lattice fits with the fuzzy principle. Thus, the interval membership degree may be thought as an approximation of the exact degree.

Let \(L\) be an bounded lattice. Thus, a t-conorm is a function \(S : IL^2 \rightarrow IL\) that is commutative, associative, monotonic and has 0 as the neutral element.

Proposition 5.1 If \(S\) is a t-conorm on the bounded lattice \(IL\) then \(\hat{S} : IL^2 \rightarrow IL\) is a t-conorm on the bounded lattice \(IL\).

Proof: It is analogous to [6, Theorem 5.1].

Since t-conorms are commutative, a characterization of \(\hat{S}\) can be expressed by:

\[
\hat{S}(X, Y) = [S(X, Y), S(X, \bar{Y})]
\]

(3)

6 Fuzzy negation on bounded lattices and the interval constructor

A function \(N : L \rightarrow L\) is a fuzzy negation on the bounded lattice \(L\) if

- N1: \(N(0) = 1\) and \(N(1) = 0\).
- N2: If \(y \leq \bot x\) then \(N(x) \leq \bot N(y)\).
In addition, fuzzy negations satisfying the involutive property are called strong fuzzy negations \([19, 8]\) on \(L\):

\[ N(\neg x) = x, \forall x \in L. \]

Let \(N : L \rightarrow L\) be a fuzzy negation. A characterization of \(\hat{N}\) is presented in the following:

\[
\hat{N}(X) = [N(\overline{X}), N(\overline{X})]. \tag{4}
\]

**Theorem 6.1** Let \(N : L \rightarrow L\) be a fuzzy negation on the bounded lattice \(L\). Then \(\hat{N}\) is a fuzzy negation on the bounded lattice \(\mathbb{L}\). In addition, if \(N\) is a strong fuzzy negation on \(L\) then \(\hat{N}\) is a strong fuzzy negation on \(\mathbb{L}\).

**Proof:** \(N1\): Trivially, \(N1\) is satisfied.

\(N2\): If \(y \leq_{IL} x\) then \(\overline{y} \leq_{IL} \overline{x}\) and \(\overline{y} \leq_{IL} x\).

Therefore, by \(N2\), \(\hat{N}(X) = [N(\overline{X}), N(\overline{X})] \leq_{IL} [N(\overline{Y}), N(\overline{Y})]\) and \(\hat{N}(X) \leq_{IL} \hat{N}(Y)\).

\(N3\): Based on the result summarized in Eq. (4), \(\hat{N}(\hat{N}(X)) = \hat{N}([N(\overline{X}), N(\overline{X})])\) and, therefore, \(\hat{N}(\hat{N}(X)) = [N(\overline{X}), N(\overline{X})]\). So, since \(N\) is involutive, \(\hat{N}(\hat{N}(X)) = X\). \(\blacksquare\)

Notice that if \(X \subseteq Y\) then \(\overline{X} \subseteq \overline{Y}\) and \(\overline{Y} \leq_{IL} \overline{X}\).

Therefore, by \(N2\),

\[ \hat{N}(X) = [N(\overline{X}), N(\overline{X})] \subseteq [N(\overline{Y}), N(\overline{Y})] \]

and \(\hat{N}(X) \subseteq \hat{N}(Y)\). Thus, a fuzzy negation \(N : L \rightarrow L\) also is inclusion monotonic function.

## 7 Fuzzy Implication

Several definitions for fuzzy implication together with related properties have been given (see \([8, 14, 27, 28]\)) for the unit interval lattice. The unique consensus in these definitions is that the fuzzy implication should have the same behavior as the classical implication for the crisp case.

In order to obtain fuzzy S-implications defined on arbitrary bonded lattice, let \(L\) be a bounded lattice. A binary function \(I : L^2 \rightarrow L\) is a fuzzy implication on \(L\) if \(I\) satisfies the minimal boundary conditions:

\[ I(1, 1) = I(0, 1) = I(0, 0) = 1 \text{ and } I(1, 0) = 0. \]

Some reasonable properties may be required for fuzzy implications. The properties considered in this paper are listed below:

- 11: If \(x \leq_{IL} z\) then \(I(x, y) \geq_{IL} I(z, y)\);
- 12: If \(y \leq_{IL} z\) then \(I(x, y) \leq_{IL} I(x, z)\);
- 13: \(I(1, x) = x\) (left neutrality principle);
- 14: \(I(x, I(y, z)) = I(y, I(x, z))\) (exchange principle).

### 7.1 S-implications and the interval constructor

Let \(S\) be a t-conorm and \(N\) be a fuzzy negation on the bounded lattice \(L\). Then a fuzzy implication called S-implication on \(L\) is given by:

\[ I_{S,N}(x, y) = S(N(x), y). \tag{5} \]

An S-implication arises from the notion of disjunction and negation using the corresponding tautology of classical logic. More specifically, S-implications are based on the classical logical equivalence: \(\alpha \rightarrow \beta \equiv \neg\alpha \vee \beta\).

One can notice that in some texts (like, e.g., in \([2, 8, 14]\)), an S-implication requires strong fuzzy negation. As the approach presented in \([19]\), in this work this condition is not required. The main results relating the S-implication and the properties I1, \ldots, I5 are presented in the following.

**Proposition 7.1** Let \(I : L^2 \rightarrow L\) be a fuzzy implication. \(I\) is an S-implication on \(L\) if and only if the properties I1, I2, I3 and I4 are satisfied.

**Proof:** It is analogous to \([2]\). \(\blacksquare\)

Considering any fuzzy implication on an arbitrary bounded lattice \(L\), it is always possible to obtain canonically an interval fuzzy implication. The interval fuzzy implication also meets the optimality property and preserves the same properties satisfied by the fuzzy implication. In the following two propositions, the best interval representation of a fuzzy implication is shown as an inclusion-monotonic function in both arguments. The related proofs are straightforward, following from the definition of \(\hat{I}\) as a particular case of the Eq. (2).

**Proposition 7.2** If \(I\) is a fuzzy implication on \(L\) then \(\hat{I}\) is a fuzzy implication on \(\mathbb{L}\).

**Proof:** It is straightforward. \(\blacksquare\)
Proposition 7.3 Let $I$ be a fuzzy implication on $L$. Then, for each $X_1, X_2, Y_1, Y_2 \in IL$, if $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$ then $I(X_1, Y_1) \subseteq I(X_2, Y_2)$.

Proof: It is straightforward.

Theorem 7.1 Let $I$ be a fuzzy implication on $L$. If $I$ satisfies a property $I_k$, for some $k = 1, \ldots, 4$, then $\hat{I}$ also satisfies the property $I_k$.

Proof: I1: If $u \in \hat{I}(X, Y)$, then there exist $x \in X$ and $y \in Y$ such that $I(x, y) = u$. If $X \leq_{IL} Z$ then there exists $z \in Z$ and $x \leq_{IL} z$. So, by I1, $u = I(x, y) \geq_{IL} I(z, y)$. On the other hand, if $v \in \hat{I}(Z, Y)$ then there exist $z \in Z$ and $y \in Y$ such that $I(z, y) = v$. If $X \leq_{IL} Z$ then $x \leq_{IL} z$, for some $x \in X$. So, by I1, $\hat{I}(x, z) \geq_{IL} I(z, y) = v$. Therefore, for each $u \in \hat{I}(X, Y)$, there is $v \in \hat{I}(Z, Y)$ such that $u \geq_{IL} v$. In addition, for each $v \in \hat{I}(Z, Y)$ there is $u \in \hat{I}(X, Y)$ such that $u \geq_{IL} v$. Hence, $\hat{I}(X, Y) \geq_{IL} \hat{I}(Z, Y)$.

I2: If $u \in \hat{I}(X, Y)$ then there exist $x \in X$ and $y \in Y$ such that $I(x, y) = u$. If $Y \leq_{IL} Z$ then there exists $z \in Z$ such that $y \leq_{IL} z$. So, by I2, $u = I(x, y) \leq_{IL} I(x, z)$. On the other hand, if $v \in \hat{I}(X, Z)$ then there exist $y \in Y$ and $x \in X$ such that $I(x, y) = v$. If $Y \leq_{IL} Z$ then $y \leq_{IL} z$, for some $y \in Y$. So, by I2, $I(x, y) \geq_{IL} I(x, z) = v$. Therefore, for each $u \in \hat{I}(X, Y)$, there is $v \in \hat{I}(X, Z)$ such that $u \leq_{IL} v$ and, for each $v \in \hat{I}(X, Z)$, there is $u \in \hat{I}(X, Y)$ such that $u \leq_{IL} v$. Hence, $\hat{I}(X, Y) \leq_{IL} \hat{I}(X, Z)$.

I3: Trivially, by I3, for each $x \in X$, $I(1, x) = x$ and so $\{I(1, x) : x \in X\} = X$. Thus, since $\hat{I}([1, 1], X)$ is the narrowest interval containing $\{I(1, x) : x \in X\}$, then $\hat{I}([1, 1], X) = X$.

I4: If $u \in \hat{I}(X, \hat{I}(Y, Z))$ then there exist $x \in X$, $y \in Y$ and $z \in Z$ such that $I(x, I(y, z)) = u$. But, by I4, $u = I(y, I(x, z))$. So, $u \in \hat{I}(Y, \hat{I}(X, Z))$ and, therefore, $I(X, \hat{I}(Y, Z)) \subseteq \hat{I}(X, \hat{I}(Y, Z))$. Analogously, if $u \in \hat{I}(Y, \hat{I}(X, Z))$ then there exist $x \in X$, $y \in Y$ and $z \in Z$ such that $I(y, I(x, z)) = u$. But, by I4, $u = I(x, I(y, z))$. So, $u \in \hat{I}(X, \hat{I}(Y, Z))$ and, therefore, $\hat{I}(Y, \hat{I}(X, Z)) \subseteq \hat{I}(X, \hat{I}(Y, Z))$. Hence, $\hat{I}(X, \hat{I}(Y, Z)) = \hat{I}(Y, \hat{I}(X, Z))$.

Proposition 7.4 Let $I : L^2 \rightarrow L$ be a fuzzy implication on a bounded lattice $L$ satisfying the properties $I_1$ and $I_2$. Then a characterization of $\hat{I}$ can be obtained as

$$\hat{I}(X, Y) = \{I(\sup(X, Y), \inf(X, Y)),$$

$$I(\inf(X, Y), \sup(X, Y))\}.$$

Proof: If $X \leq_{IL} X$ and $Y \leq_{IL} Y$ then $\inf(X, Y) \leq_{IL} X, Y \leq_{IL} \sup(X, Y)$. By the properties $I_1$ and $I_2$, $I(\sup(X, Y), \inf(X, Y)) \leq_{IL} I(x, y) \leq_{IL} I(\inf(X, Y), \sup(X, Y))$. Therefore, $\hat{I}(X, Y) \subseteq I(\sup(X, Y), \inf(X, Y))$. If $\hat{I}(X, Y)$ are lower and upper bounds of $\hat{I}$, respectively. Hence, because $I(\inf(X, Y), \sup(X, Y))$ belongs to $\hat{I}$, they are the infimum and supremum of $\hat{I}$.

7.2 S-implications on bounded lattices and the interval constructor

Theorem 7.2 Let $S$ be a t-conorm on $L$ and $N$ be a fuzzy negation on $L$. Then it holds that $\hat{I}_{\hat{S}, \hat{N}} = \hat{S} \hat{N}$.

Proof: Consider $X, Y \in IL$. Applying the results of Eq.(4), one has that $\hat{I}_{\hat{S}, \hat{N}}(X, Y) = \hat{S} \hat{N}(X, Y)$. Thus, by Eq.(3), $\hat{I}_{\hat{S}, \hat{N}}(X, Y) = \hat{S}([N(X), N(X)], Y)$. Based on the definition of a fuzzy implication, presented in Eq.5, it follows that $[I_{\hat{S} \hat{N}}(X, Y), I_{\hat{S} \hat{N}}(X, Y)]$. Finally, by definition of the best representation of the function $I_{\hat{S} \hat{N}}$, given in Eq. 2, it follows that $\hat{I}_{\hat{S}, \hat{N}}(X, Y) = \hat{I}_{\hat{S}, \hat{N}}(X, Y)$.

The next corollary follows directly.

Corollary 7.1 If $I$ is an S-implication on $L$ then $\hat{I}$ is an S-implication on $\hat{P}L$.

The above results together with Theorem 7.2 state the commutativity of the diagram in Fig. 1, where $C(S)_L$ denotes the class of t-conorms on $L$, $C(N)_L$ indicates the class of fuzzy negations on $L$ and $C(I)_L$ is the class of S-implications on $L$.

8 Final Remarks

The emphasis on bounded lattice applied to both interval mathematics and fuzzy set theory are firmly integrated with principles of information theory used to underlie logic systems for expert systems.
\[ C(S)_L \times C(N)_L \xrightarrow{eq(5)} C(I)_L \]
\[ (eq(3), eq(4)) \]
\[ C(S)_IL \times C(N)_IL \xrightarrow{eq(5)} C(I)_IL \]

Figure 1: Commutative diagram relating S-implication on L with S-implications on IL

This paper may contribute to extend the fuzzy S-implication notion for the more general framework of bounded lattices and provide an interval constructor which preserves the good properties of that connective. The interval constructor considers the best interval representation of a fuzzy connective on arbitrary bounded lattices to deal with the imprecision of a specialist in providing an exact value to measure membership uncertainty. Thus, this paper complements the results of previous works [5, 6, 7].

In this work, we mainly discussed under which conditions generalized fuzzy S-implications applied to interval values preserve properties of canonical forms generated by interval t-conorms. It was shown that properties of fuzzy logic may be naturally extended for interval fuzzy degrees considering the respective degenerate intervals.

Since implication in fuzzy systems may be used to deal with rules which play an important role in such systems, the results presented in this paper are important not only to analyze deductive systems in mathematical depth but also to provide foundations of methods based on interval fuzzy logic.

They integrate two important features: the accuracy criteria and the optimality property of interval computations, and a formal mathematical theory for the representation of uncertainty, concerned with the fuzzy set theory. The former gives a more reliable modelling of real systems and the latter is crucial for their management and control. Thus they can be useful to provide a mathematical foundation for interval fuzzy systems such as the one presented in [4].

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References


