Maximal Attractor for a Ostwald Ripening Model

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We consider a family of phase-field models that couples a Cahn-Hilliard with several Allen-Cahn equations. We show that such family of phase-field models possesses a maximal attractor with finite fractal dimension.

Ostwald ripening is a phenomenon observed in a wide variety of two-phase systems in which there is coarsening of one phase dispersed in the matrix of another. Because its practical importance, this process has been extensively studied in several degrees of generality. In particular for Ostwald ripening of anisotropic crystals, Fan et al. [10] presented a model taking in consideration both the evolution of the compositional field and of the crystallographic orientations. In the work of Fan et al. [10], there are also numerical experiments used to validate the model.

By defining orientation and composition field variables, the kinetics of coupled grain growth can be described by their spatial and temporal evolution, which is related with the total free energy of the system. The microstructural evolution of Ostwald ripening can be described by the Cahn-Hilliard/Allen-Cahn system:

\[
\begin{aligned}
\partial_t c &= \nabla \cdot [D \nabla (\partial_c F - \kappa_c \Delta c)], \\
\partial_t \theta_i &= -L_i (\partial_{\theta_i} F - \kappa_i \Delta \theta_i),
\end{aligned}
\]

in \( \Omega_T \) and

\[
\partial_n c = \partial_n (\partial_c F - \kappa_c \Delta c) = \partial_n \theta_i = 0,
\]

for \( x \in \Omega \), where \( i = 1, \ldots, p \).

Here, \( \Omega \) is the physical region where the Ostwald process is occurring; \( \Omega_T = \Omega \times (0, T) \); \( S_T = \partial \Omega \times (0, T) \); \( 0 < T < +\infty \); \( n \) denotes the unitary exterior normal vector and \( \partial_n \) is the exterior normal derivative at the boundary; \( c(x, t) \), for \( t \in [0, T] \), \( 0 < T < +\infty \), \( x \in \Omega \), is the compositional field (fraction of the soluto with respect to the mixture); \( \theta_i(x, t) \), for \( i = 1, \ldots, p \), are the crystallographic orientations fields; \( D, \lambda_c, L_i, \lambda_i \) are positive constants related to the material properties.

In this paper, we consider a family of phase-field models related to that presented by [10]. As in [10], it is assumed that the local free energy \( F \) has the following form:

\[
\begin{aligned}
F(c, \theta_1, \ldots, \theta_p) &= -\frac{A}{2} (c - c_\mu)^2 + \frac{B}{4} (c - c_\mu)^4 \\
&+ \frac{D_\alpha}{4} (c - c_\alpha)^4 + \frac{D_\beta}{4} (c - c_\beta)^4 - \gamma \sum_{i=1}^p g(c, \theta_i) \\
&+ \frac{\delta}{4} \sum_{i=1}^p \theta_i^4 + \sum_{i=1}^p \sum_{i \neq j=1}^p \varepsilon_{ij} f(\theta_i, \theta_j).
\end{aligned}
\]

\( A, B, D_\alpha, D_\beta, \gamma, \delta, \varepsilon_{ij}, i \neq j = 1, \ldots, p \) are positive constants related to the material properties, \( c_\alpha \) and \( c_\beta \) are the solubilities or equilibrium concentrations for the matrix phase and second phase, respectively, and

\[
c_\mu = (c_\alpha + c_\beta)/2.
\]

Functions \( f \) and \( g \) are assumed to satisfy the following properties:

\[
f, g \in C^2(\mathbb{R}^2, \mathbb{R}),
\]
|f(a, b) − f(u, v) + ∇f(u, v) · (u − a, v − b)|
≤ F_1(a − u)^2 + F_2(v − b)^2
(≤ \max\{F_1, F_2\}|(u, v) − (a, b)|^2)\)

and
\[|g(a, b) − g(u, v) + ∇g(u, v) · (u − a, v − b)|\]
\[≤ G_1(a − u)^2 + G_2(v − b)^2\]

for all \((u, v), (a, b) ∈ \mathbb{R}^2\) and fixed constants \(F_1, F_2, G_1, G_2 ≥ 0\).
We remark that the previous assumptions on the functions \(f\) and \(g\) imply
that the difference between \(f(a, b)\) and \(g(a, b)\) and their Taylor polynomials of degree one at
\((u, v)\), respectively, are bounded up to a multiplicative fixed constant by the square of
the Euclidean distance between \((u, v)\) and \((a, b)\).

We also remark that the local free energy \(\mathcal{F}\) is assumed to have form like the one stated
above in order to comply to a requirement of Chen et al. [2] – [5] that it should have 2p
degenerate minima at the equilibrium concentration \(c_0\) to distinguish the 2p orientations differences
of the second phase grains in space.

In this paper, we show that such family of phase-field models satisfying Problem (1) pos-
sesses a maximal attractor. Our approach to the problem is to associate with Problem (1)
a semigroup that satisfies the usual semigroup properties and to show the existence of
appropriate absorbing sets. With such facts, we may conclude that the semigroup possesses a maxi-
mal attractor. Since we also prove the existence of inertial sets which have finite fractal, we
obtain an upper bound for the fractal dimension of the attractor.

An essential remark for the study of Problem (1) is that the space integral of \(c\) is con-
served in time, which we express in the form
\[\langle c(t), 1 \rangle_{(H^1)'_\Omega} = \langle c_0, 1 \rangle_{(H^1)'_\Omega}, \ \forall t > 0.\]

We also note that Problem (1) has the following Lyapunov functional \(J(c, \theta_1, \ldots, \theta_p)\) given by:
\[
\int_{\Omega} \left( \mathcal{F}(c, \theta_1, \ldots, \theta_p) + \frac{\kappa_c}{2D} |\nabla c|^2 + \sum_{i=1}^{p} \frac{\kappa_i}{2L_i} \theta_i^2 \right).
\]

These two fact will be essential tools in the proofs.

For our analysis of Problem (1), we introduce the spaces:
\[H = (H^1(\Omega))^p \times [L^2(\Omega)]^p\]
and \(V\) the space of functions such that
\[V ⊂ H^2(\Omega) \times [H^1(\Omega)]^p.\]

For \((c, \theta_1, \ldots, \theta_p) \in V\), we have \(\partial_\nu c = 0\) em \(\partial \Omega\).

It is also convenient to introduce the functions spaces
\[H_\rho = \left\{ (c, \theta_1, \ldots, \theta_p) ∈ H; \frac{1}{|\Omega|} \int_{\Omega} (c, 1)_{(H^1)'_\Omega} = \rho \right\},\]
\[\bar{H}_\rho = \left\{ c ∈ L^2(\Omega); \frac{1}{|\Omega|} \int_{\Omega} c = \rho \right\}\]
and
\[\mathcal{H}_R = \bigcup_{|\rho| ≤ R} H_\rho.\]

Our paper is sketched as follows:

**Preliminary Results.** For the mathematical study of Problem (1) it is useful to consider two
equivalent definitions for the inner product in \((H^1(\Omega))^p\). We denote by
\[0 = \lambda_1 < \lambda_2 ≤ \cdots ≤ \lambda_k ≤ \cdots\]
the eigenvalues of the operator \(−Δ\) with ho-

mogeneous Neumann boundary conditions and by \(w_k, k = 1, \ldots,\), the corresponding eigen-
functions such that \(|w_k|_{L^2(\Omega)} = 1, k = 1, \ldots,\). The \(\{w_k\}\) are a complete orthonormal family in
\(L^2(\Omega)\) as well as a complete orthogonal family in \(H^1(\Omega)\). Next we define the following scalar product in \((H^1(\Omega))^p)\:
\[\langle (u, v)_{-1}, (u, w) \rangle = \langle u, w \rangle_1 \langle v, w \rangle_1 \]
+ ∞ \sum_{j=2}^\infty \frac{1}{\lambda_j} \langle u, w_j \rangle \langle v, w_j \rangle \tag{2}\]

where the notation \(\langle \cdot, \cdot \rangle\) is used to denote the duality product between \(H^1(\Omega)\) and \((H^1(\Omega))^p)\).
In what follows, we shall sometimes use the notation for the inner product in \(L^2(\Omega)\).

We deduce from the density of \(L^2(\Omega)\) in \((H^1(\Omega))^p\) and from (2) that the \(\{w_k\}\) are a com-
plete orthogonal family in \((H^1(\Omega))^p\). Note that we also have for all \(u ∈ (H^1(\Omega))^p)\)
\[u = (u, w_1)_{-1} w_1 + \sum_{j=2}^\infty \lambda_j (u, w_j)_{-1} w_j\]
\[= \sum_{j=1}^\infty (u, w_j) w_j\]
and that
\[ \langle w_i, w_j \rangle = \int_\Omega w_i w_j = \delta_{ij}, \quad i, j = 1, \ldots \]

For \( u \in (H^1(\Omega))' \), we define
\[ m(u) = \frac{1}{|\Omega|} \langle u, 1 \rangle \]
and
\[ \overline{u} = u - m(u) \]
and give an alternative definition for the scalar product in \((H^1(\Omega))'\). For \( u \in (H^1(\Omega))' \), let
\[ \psi = Nu \]
be the unique solution in \( H^1(\Omega) \) in the sense of distributions of the problem
\[
\begin{cases}
-\Delta \psi = \overline{u} \\
\partial_t \psi = 0 \\
\int_\Omega \psi(x) dx = 0
\end{cases}
\]

If \( u, v \in (H^1(\Omega))' \) and if \( \psi = Nu \) and \( \chi = Nv \), then
\[ (u, v)_1 = \frac{1}{|\Omega|} \langle u, 1 \rangle \langle v, 1 \rangle + \int_\Omega \nabla \psi \nabla \chi \]
and
\[ \| u \|_1^2 = |\Omega| (m(u))^2 + \int_\Omega |\nabla \psi|^2. \]

1. **Galerkin Method.** We use the Galerkin method, to show that for any \((c_0, \theta_{10}, \ldots, \theta_{p0}) \in H_\rho\), Problem (1) has a unique solution \((c, \theta_1, \ldots, \theta_p)\) which satisfies
\[
(c, \theta_1, \ldots, \theta_p) \in L^\infty(0, T; H_\rho)
\]
\[
(c, \theta_1, \ldots, \theta_p) \in L^2(0, T; (H^1(\Omega))^{p+1}),
\]
\[ c \in L^2(Q_T) \]
for all \( T > 0 \), where \( Q_T = \Omega \times (0, T) \) and \((c, \theta_1, \ldots, \theta_p) \in C(\mathbb{R}^+; H_\rho)\). If furthermore \[ c_0 \in \overline{H}_\rho, \] then
\[ c \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega)). \]

We also show that if
\[ (c_0, \theta_{10}, \ldots, \theta_{p0}) \in V \cap \overline{H}_\rho \]
then
\[
(c, \theta_1, \ldots, \theta_p) \in L^\infty(0, T; (H^1(\Omega))^{p+1})
\]
\[
(c, \theta_1, \ldots, \theta_p) \in L^2(0, T; (H^2(\Omega))^{p+1}),
\]
\[ \Delta c \in L^2(0, T; H^1(\Omega)),
\]
\[ (\partial_t c, \partial_t \theta_1, \ldots, \partial_t \theta_p) \in (L^2(Q_T))^{p+1} \]
We prove the mapping
\[ S(t) : (c_0, \theta_{10}, \ldots, \theta_{p0}) \mapsto (c(t), \theta_1(t), \ldots, \theta_p(t)) \]
is Hölder continuous with exponent \( \frac{1}{2} \) on \( H \) for all \( t > 0 \) and \( \{S(t)\}_{t \geq 0} \) is a semigroup on \( \mathcal{H}_R \).

Finally we show the functional \( J(c(t), \theta_1(t), \ldots, \theta_p(t)) \) decays along orbits so that it is a **Lyapunov functional** for Problem (1).

All the results above are obtained by considering some approximate solutions: for each integer \( m \), we look for a solution \((c_m, \theta_{1m}, \ldots, \theta_{pm})\) of the form
\[ c_m(t) = m_0 \sqrt{|\Omega|} w_1 + \sum_{j=2}^m c_{jm}(t) w_j \]
and
\[ \theta_{im} = \sum_{j=1}^m \theta^i_{jm}(t) w_j \]
satisfying
\[
\int_\Omega \partial_t c_m w_j + D \int_\Omega (\kappa \Delta c_m - \partial_v F_m) \Delta w_j = 0
\]
\[
\int_\Omega \partial_t \theta_{im} w_j + L_i \int_\Omega (\partial_v F_m w_j + \kappa_i \nabla \theta_{im} \nabla w_j) = 0
\]
for \( j = 1, \ldots, m, \)
\[ c_m(0) = \sum_{j=1}^m (c_0, w_j) w_j \]
and
\[ \theta_{im}(0) = \sum_{j=1}^m (\theta_{10}, w_j) w_j \]
where \( F_m = F(c_m, \theta_{1m}, \ldots, \theta_{pm}) \).

The problem above is an initial value problem for a system of \((p + 1)m\) ordinary differential equations, so that it has a unique
solution \((c_m, \theta_1, \ldots, \theta_m)\) on some interval \((0, T_m), T_m > 0\); in fact with the a priori estimates we deduce, we can show that \(T_m = +\infty\).

To prove the uniqueness of solution, we use ideas of Elliott and Luckhaus [9].

2. Absorbing Sets. We show the existence of bounded absorbing sets in \(\mathcal{H}_\alpha\) and in \((H^1(\Omega))^{p+1} \cap \mathcal{H}_\alpha\) for the semigroup associated with Problem 1 so that \(S(t)\) possesses a maximal attractor in \(\mathcal{H}_\alpha\).

The existence of the absorbing set are consequence of the \textit{a priori} estimates and of the application of the Uniform Gronwall Lemma. In the first part we argue as Brochet et al. [1].

2.1 We prove that for any \(\alpha \geq 0\), there exists a constant \(R = R(\alpha) > 0\) such that \(B_{\mathcal{H}_\alpha}(0, R(\alpha))\) is an absorbing set for \(S(t)\) in \(\mathcal{H}_\alpha\).

2.2 We show the existence of an absorbing set in \((H^1(\Omega))^{p+1} \cap \mathcal{H}_\alpha\).

2.3 We deduce from Temam [11] that for every \(\alpha \geq 0\) the semigroup \(S(t)\) associated with Problem (1) maps \(\mathcal{H}_\alpha\) into itself. It possesses in \(\mathcal{H}_\alpha\) a maximal attractor \(\mathcal{A}_\alpha\) that is connected.

2.4 We argue as Cherfils and Miranville [6] and Debussche and Dettori [7] to prove that \((c, \theta_1, \ldots, \theta_p)\) entering an absorbing set of \((H^2(\Omega))^{p+1} \cap \mathcal{H}_\alpha\).

3. Inertial Sets. We prove the existence of inertial sets, namely, compact sets which contain the attractor, which are positively invariant by the semigroup, which have finite fractal dimension and which attract all solutions at an exponential rate so that we also obtain an upper bound for the fractal dimension of the attractor.

The key arguments to prove the existence of an inertial set is to verify the \textit{squeezing property} and to show that the semigroup \(S(t)\) satisfies a Lipschitz property. We show the existence of inertial sets for the semigroup \(S(t)\) corresponding to Problem 1 on \(\mathcal{H}_\alpha\) where \(\alpha\) is any given nonnegative constant.

We consider \(B_\alpha\) the absorbing ball in \((H^2(\Omega))^{p+1} \cap \mathcal{H}_\alpha\) and define the positively invariant set

\[
X_\alpha = \bigcup_{t \geq t_0} S(t)B_\alpha, \quad t_0 = 2 + t_0(R(\alpha)).
\]

We remark that with the results of the previous steps we have that \(X_\alpha\) is bounded in \((C(\overline{\Omega}))^{p+1}\).

3.1 We prove the semigroup \(S(t)\) satisfies a Lipschitz property from \(X_\alpha\) into itself.

3.2 (The squeezing property) We show that the semigroup \(S(t)\) corresponding to Problem (1) satisfies the squeezing property:

for any \(t^* > 0\), there exists \(N_0 = N_0(t^*)\) such that for any \(U_1 = (c^1, \theta^1, \ldots, \theta^p)\) and \(U_2 = (c^2, \theta^2, \ldots, \theta^p)\) in \(X_\alpha\) satisfying

\[
\|P_{N_0}(S(t^*)U_1 - S(t^*)U_2)\|_H \leq \|(I - P_{N_0})(S(t^*)U_1 - S(t^*)U_2)\|_H
\]

one has the inequality

\[
\|S(t^*)U_1 - S(t^*)U_2\|_H \leq \frac{1}{8} \|U_1 - U_2\|_H.
\]

3.3 We apply Theorem 2.1 of Eden et al. [8] to obtain, for \(X_\alpha\), an inertial set \(\mathcal{M}_\alpha\) for \((S(t))_{t \geq 0}, X_\alpha\) which has fractal dimension \(\leq \text{constant} \cdot N_0\).

Suppose that \(N_0\) is large. We have that (see [11], p. 300)

\[
\lambda_{N_0} \sim C|\Omega|^{-\frac{2}{n}}N_0^{\frac{2}{n}}.
\]

For large \(N_0\), we may choose \(N_0\) such that

\[
\lambda_{N_0} \leq C(t^*) \leq \lambda_{N_0 + 1}.
\]

Then

\[
C|\Omega|^{-\frac{2}{n}}N_0^{\frac{2}{n}} \leq 2C(t^*)
\]

and thus

\[
N_0 \leq \left(\frac{2C(t^*)}{C}\right)^\frac{n}{2} |\Omega|
\]

so that

\[
\text{fractal dimension} \leq \text{constant} \cdot |\Omega|
\]
References


