Numerical Methods for Generalized KdV equations

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Resumo: In this paper we study a class of fully discrete scheme for the generalized Korteweg-de Vries equation in a bounded domain \((0, L)\). Taking a particular discretization for the nonlinear term when \(p = 4\) (that is the critical case of the generalized KdV equation) we obtain thanks to an estimate in \(L^2(0, T; H^1(0, L))\) and the convergence in \(L^4\)-strong for the solution of the numerical scheme. We provide several numerical results to illustrate the performance of the method.

1 Introduction

In this work we study a class of fully discrete scheme for the critical generalized Korteweg-de Vries equation. Taking a particular discretization for the nonlinear term when \(p = 4\) (that is the critical case of the generalized KdV equation) we obtain thanks to an estimate in \(L^2(0, T; H^1(0, L))\) for the solution of the numerical scheme, the convergence.

We consider the following initial boundary value problem

\[
\begin{align*}
    u_t + u_{xxx} + u^p u_x + u_x &= 0, \\
    u(0, t) &= u(L, t) = 0, \\
    u_x(L, t) &= 0, \\
    u(x, 0) &= u_0(x),
\end{align*}
\]

with \(x \in (0, L)\) and \(t > 0\). We are specifically interested in the critical case \(p = 4\). The equation (1) corresponds to a modified form of the Generalized Korteweg-de Vries equation (GKdV equation henceforth)

\[
    u_t + u_{xxx} + u^p u_x = 0,
\]

where we added a linear convective term \(u_x\) due to the boundary conditions. The generalized Korteweg-de Vries equation (5) has been extensively studied for understanding the interaction between the dispersive term and the nonlinearity in the context of the theory of nonlinear dispersive evolution equations. Several authors proving stability properties of the solutions when \(p < 4\) (see Bona et al. [5] and the reference therein). On the other hand, numerical simulation indicate that for \(p \geq 4\), smooth solution of the initial-value problem may form singularities in finite time (see [3, 4]). When \(p = 4\), the equation (1) considered in real line is called critical for various reason. On of these reason is that the mass remains invariant by scaling in the \(L^2\)-norm. Local results for data in \(L^2\) were established by Kenig, Ponce and Vega [12]. On the other hand, in [14] Merle showed that \(H^1\) solutions may blow-up in finite time. Thus the nonlinearity is critical for the long time behavior of solutions.

The novelty of the present work is to introduce a numerical scheme preserving the asymptotical behaviour property of exponential decay with a rate independent of the mesh size, we discretize the nonlinear term of the equation in a particular way, taking into account the invariance of the mass scaling in the \(L^2\)-norm.

We establish a series of estimates for the continuous problem (1)-(4) that will be used for the numerical scheme in the next section. We start by stating the following existence result due to Faminskii [10]

**Theorem 1 (See [10], Theorem 1).**
Let \(u_0 \in L^2(0, L)\) and \(T > 0\) be given. Then, there exists a \(T^* \in (0, T]\) such that the problem (1)-(4) admits a unique solution \(u \in C([0, T^*: L^2(0, L)] \cap L^2([0, T^\ast) : H^1_0(0, L))).

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On the other hand, the following estimate results is proven in [13]

**Proposition 1.** Let \( u \) be the solution of problem (1)-(4) obtained in Theorem 1. If \( \| u_0 \|_{L^2(0, L)} < \sqrt{\frac{3}{2}} \), then

\[
\| u \|_{L^2(0, T; H^1_0(0, L))} \leq c \left( 1 - \frac{4}{9} \right) \| u_0 \|_{L^2(0, L)}
\]

where \( c = c(T, L) \). Furthermore,

\[
u_t \in H^{6/5}(0, T : H^{-2}(0, L)).
\]

We start first with a description of the scheme, the wellposedness, some estimate results and the convergence of the scheme.

### 2 Description of the numerical scheme

We consider finite differences based on the unconditionally stable schemes described in [9, 18]. Let the discrete space :

\[ X_J = \{ u = (u_0, u_1, \ldots, u_J) \in \mathbb{R}^{J+1} \mid u_0 = 0 \text{ and } u_J = u_{J-1} = 0 \} \]

and

\[ (D^+ u)_j = \frac{u_{j+1} - u_j}{\delta x}, \quad (D^- u)_j = \frac{u_j - u_{j-1}}{\delta x}, \]

for \( j = 1, \ldots, J - 1 \), and \( D = \frac{1}{2}(D^+ + D^-) \) the classical difference operators, where \( \delta x \) is the space-step and \( \delta t \) is the time-step, for \( j = 0, \ldots, J \), and \( n = 0, \ldots, N \).

#### 2.1 The semidiscrete numerical scheme

We describe first a semi-discrete numerical scheme. We note by \( u_j(t) \) the approximate value of \( u(j \delta x, t) \), solutions of the nonlinear problem (1)-(4). The approximation of the nonlinear problem (1)-(4) which reads as system of EDOs :

\[
\begin{align*}
\frac{d}{dt}[u_j] + (A^{(\theta)}u)_j + F(u)_j &= 0, \quad j = 1, \ldots, J - 1, \\
u_0(t) &= u_j(t) = u_{J-1}(t) = 0, \\
u(0) &= \int_{x_j - \frac{1}{2}}^{x_j + \frac{1}{2}} u_0(x)dx,
\end{align*}
\]

with \( j = 1, \ldots, J - 1 \), where \( x_{j+\frac{1}{2}} = j + \frac{1}{2}\delta x \) and \( x_j = j\delta x \). The matrix \( A^{(\theta)} \in \mathbb{R}^{(J-1) \times (J-1)} \) is an approximation of order \( \theta \) of the dispersive term \( u_{xxxx} \) and the linear convective term \( u_x \).

For instance, if we want an approximation of 1st order in space, we can choose

\[
A^{(1)} = D^+ D^- + D.
\]

This forward differences approximation in the definition of \( A^{(1)} \) is do it in order to have a positive defined matrix \( I + \delta t A^{(1)} \) (see for instance [2, 8, 9, 18]). On the other hand, if we want an approximation of second order we can take a central differences approximation as follows

\[
A^{(2)} = D^+ DD^- + D.
\]

We obtain also a positive defined matrix for the matrix \( I + \delta t A^{(2)} \), with \( A^{(2)} \) defined in (12).

The nonlinearity \( u^4 u_x \) of the equation (1), is approached by a nonlinear function \( F(u^n) \), where \( F : \mathbb{R}^{J-1} \rightarrow \mathbb{R}^{J-1} \) is given by the following expression

\[
F(u)_j = u_j^4 (Du)_j - \frac{5}{2} u_j^3 (Du^2)_j + \frac{10}{3} u_j^2 (Du^3)_j - \frac{5}{2} u_j (Du^4)_j + \left(Du^5\right)_j
\]

for all \( j = 1, \ldots, J - 1 \).

#### Remark.

1. Using classical finite difference, \( F(u)_j \) is an approximation of

\[
\begin{align*}
u^4 u_x &= \frac{5}{2} u^{3}_x \left(u^2\right)_x + \frac{10}{3} u^2 \left(u^3\right)_x \\
&\quad - \frac{5}{2} u \left(u^4\right)_x + \left(u^5\right)_x = u^4 u_x
\end{align*}
\]

which are two ways to write the same term. We will see that using the approximation \( F(u)_j \) given by (13) we obtain \( H^1_0 \)–estimates for the solution of this numerical scheme similar to the continuous case.

2. The approximation (13) use central differences which it is a second-order approximation of \( u^4 u_x \). Therefore, it motivates to use a full second-order scheme choosing the central differences (12) as approximation of the dispersive and the linear convective terms. On the other hand,
the second-order scheme has less numerical diffusion that the first-order scheme, with what it is not possible to control some nonlinear terms in the approximation of (13) and get to the limit as we do it for the first-order scheme. In compensation, we can add a very small diffusive term in order to obtain the convergence of the method.

2.2 Full discrete implicit numerical schemes.

In order to discretize in temporal variable the system (8)-(10) of ordinary differential equations we choose implicit schemes. We note by $u^n_j$ the approximate value of $u_j(n\delta t)$, solutions of the nonlinear system (8)-(10), for $n = 0, \ldots, N$.

2.2.1 Implicit Euler.

The simplest implicit numerical scheme is given by the implicit Euler of order one, and reads as following

\begin{alignat}{2}
\frac{u_{j}^{n+1} - u_{j}^{n}}{\delta t} + (A^{(1)}u_{j}^{n+1})_{j} + F(u_{j}^{n+1})_{j} &= 0, \\
u_{0}^{n} &= u_{J}^{n} = u_{J-1}^{n} = 0, \\
u_{0}^{0} &= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_{0}(x)dx,
\end{alignat}

with $j = 1, \ldots, J - 1$, where $A^{(1)}$ is defined in (11), and $F(\cdot)$ is defined in (13).

2.2.2 Implicit Runge-Kutta of order 2.

From a practical point of view we need a more performant discretization for our numerical examples. Therefore, we consider here a temporal discretization by the 2-stage Gauss-Legendre implicit Runge-Kutta method, which correspond to the table (see [7])

\begin{alignat*}{3}
& a_{11} & a_{12} & \tau_{1} \\
& a_{21} & a_{22} & \tau_{2} \\
& b_{1} & b_{2} \\
& \frac{1}{3} & \frac{1}{3} & \frac{2\sqrt{3}}{3} & \frac{1}{2} \\
& \frac{1}{6} & \frac{1}{6} & \frac{1}{2} & \frac{1}{2} \\
& \frac{1}{2} & \frac{1}{2} & \frac{2\sqrt{3}}{3} & \frac{1}{3} \\
& \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\
\end{alignat*}

\begin{alignat}{2}
\frac{2}{3}a_{\ell,m} &+ \sum_{\ell=1}^{2} \delta_{\ell,m} \left(A^{(2)}u_{n,m}^{j}\right)_{j} + F(u_{n,m}^{j})_{j} = 0, \\
\ell &= 1, 2, \\
u_{0}^{n} &= u_{J}^{n} = u_{J-1}^{n} = 0, \quad \ell = 1, 2.
\end{alignat}

The numerical scheme is now specified step by step for $n = 0, \ldots, N$. We seek $u_{j}^{n}$, by way of the intermediate stages $u_{j}^{n,\ell}$, for $\ell = 1, 2$ which are solution of the $2(J-1) \times 2(J-1)$ system of nonlinear equations

\begin{alignat}{2}
u_{j}^{n,\ell} - u_{j}^{n} &+ \sum_{m=1}^{J} a_{\ell,m} \left(A^{(2)}u_{n,m}^{j}\right)_{j} + F(u_{n,m}^{j})_{j} = 0, \\
\ell &= 1, 2. \\
u_{j}^{n+1} &= u_{j}^{n} \\
-\delta t \sum_{\ell=1}^{2} b_{\ell} \left(A^{(2)}u_{n,\ell}^{j}\right)_{j} + F(u_{n,\ell}^{j})_{j} = 0.
\end{alignat}

\begin{alignat}{2}
& a^{2} - b^{2} + (a - b)^{2} = 2(a - b)a \\
& a^{6} - b^{6} + (a - b)^{6} = 6(a^{5} - b^{5})a \\
& \quad - 15(a^{4} - b^{4})a^{2} \\
& \quad + 20(a^{3} - b^{3})a^{3} \\
& \quad - 15(a^{2} - b^{2})a^{4} \\
& \quad + 6(a - b)a^{5}
\end{alignat}

Notation. We note the following internal product in $X_{J}$

\begin{alignat}{2}
(z, w) &= \sum_{j=1}^{J} \delta x z_{j} w_{j},
\end{alignat}
\((z, w)_x = (z, xw) = \sum_{j=1}^{J-1} j \delta x^2 z_j w_j\)

for all \(z, w \in \mathbb{R}^{J+1}\), and the norms: 
\(|z|_\infty = \sqrt{\langle z, z \rangle}, \quad |z|_x = \sqrt{\langle z, z \rangle_x}\), for all \(z \in \mathbb{R}^{J+1}\). Additionally, we note the \(p\)-norms in \(X_j\) defined by:

\[
|z|_p = \left( \sum_{j=1}^{J-1} \delta x |z_j|^p \right)^{1/p},
\]

\[
|z|_\infty = \max_{j=1, \ldots, J-1} |z_j|,
\]

for all \(z, w \in \mathbb{R}^{J+1}\).

First we give some standard identities introduced in [9] and used in [2, 8, 18].

**Lemma 1.** For all \(z, w \in X_j\), using the Notation 2.2.2, we have the following identities

\[
(D^+ z, w) = -(z, D^- w) \quad (23)
\]

\[
(D^+ z, z)_x = \frac{1}{2} \left( [\langle z, z \rangle_1] + \delta x |D^+ z|_2^2 \right) \quad (24)
\]

\[
(D^+ z, w)_x = -(z, D^- w)_x + \delta x (z, D^- w) - (z, w), \quad (25)
\]

\[
(D^+ D^+ D^- z, z) = \frac{1}{2} \left[ \langle (D^- z) \rangle_1 \right]^2 + \frac{\delta x}{2} |D^+ D^- z|^2, \quad (26)
\]

\[
(D^+ D^+ D^- z, z)_x = - \frac{\delta x}{2} \langle (D^- z) \rangle_1 + 3 \frac{\delta x}{2} |D^+ D^- z|^2
\]

\[
+ \frac{\delta x}{2} |D^+ D^- z|_2^2 - \frac{\delta x}{2} |D^+ D^- z|^2 \quad (27)
\]

**Proof.** Using the identity (10) with \(a = z_j, w_j\) and \(b = z_{j-1}, w_{j-1}\), multiplying by 1 and \(j \delta x\), and summing by parts over \(j = 1, \ldots, J-1\), we obtain (23)-(28). \(\blacksquare\)

On the other hand, the estimate for the nonlinear terms is given by the choice of (13) and the following lemma,

**Lemma 2.** For all \(u \in X_j\), we have

\[
(u, F(u)) = 0, \quad (29)
\]

\[
(u, F(u))_x = - \frac{1}{6} |u|^6, \quad (30)
\]

where \(F(u)\) is defined by (13).

**Proof.** Using the identity (21) with \(a = u, b = u, \ldots, J-1\), and summing over \(j = 1, \ldots, J-1\), we obtain (29) with algebraic arguments. Repeating the same argument multiplying by \(j \delta x\) before to summing over \(j = 1, \ldots, J-1\), we deduce (30). \(\blacksquare\)

We suppose that we can solve (8)-(10). Then, the estimate results of Lemma 1 and Lemma 2 give the following result of stability for the solution of the numerical scheme:

**Proposition 2.** Let \((u^n)_{n \in \mathbb{N}}\) a sequence in \(X_j\) built by the numerical scheme (8)-(10), with \(A\) and \(F(u^n)\) defined by (11)-(13), and let be \(T = n \delta t\). If \(|u^0|_2 \leq \sqrt{2}\), then, there exist a constant \(C > 0\) independent of \(\delta x\) and \(\delta t\), such that

\[
|u^n|_2 \leq |u^0|_2 \quad (31)
\]

\[
\sum_{k=0}^{n} \delta t \left| D^- u^k \right|_2 \leq C \frac{|u^0|_2^2}{1 - \frac{4}{9} |u^0|_2^4} \quad (32)
\]

\[
|D^+ D^- u^n_3|_2 \leq \frac{\delta x^{-1/2}}{\sqrt{2}} |u^0|_2 \quad (33)
\]

**Proof.** First, we multiply the equation (8) by \(u^{n+1}\) and we sum over \(j = 1, \ldots, J-1\), using (23), (24), (27) and (29). Thus multiplying by \(\delta t\) and summing over \(k = 0, \ldots, n\) using the identity (21) with \(a = u^n\) and \(b = u^{n+1}\), we obtain (31) and (33).

On the other hand, we multiply the equation (8) by \(j \delta x u^{n+1}\) and we sum over \(j = 1, \ldots, J-1\), using (25), (26), (28) and (30). Thus multiplying by \(\delta t\) and summing over \(k = 0, \ldots, n\) using the identity (10) with \(a = u^n\) and \(b = u^{n+1}\), we obtain

\[
\sum_{k=0}^{n} \delta t \left| D^- u^k \right|_2^2 \leq \quad (34)
\]

\[
\frac{(T + L + \delta x)}{3} |u^0|_2^2 + \frac{1}{9} \sum_{k=0}^{n} \delta t |u^k|_6^6.
\]

The rest of the proof is a discrete version of the proof of Proposition 1 (see [13]): it is not difficult to prove the following discrete Gagliardo-Nirenberg type inequalities (see [15]):

\[
|u|_2^2 \leq 2 |u|_2 |Du|_2 \quad (35)
\]

\[
|u|_4^4 \leq 2 |u|_2^3 |Du|_2 \quad (36)
\]
the second term of \((34)\) can be estimated by the inequalities \((35), (36),\) and \((31)\) as follow
\[
\sum_{k=0}^{n} \delta t |u^k[0]_0|^2 \leq 4 |u_0|^4 \sum_{k=0}^{n} \delta t |Du^k|_2^2.
\] (37)
Replacing \((37)\) in \((34),\) and using the fact that \(|Du|_2^2 \leq |D- u|^2_2\) (by triangular inequality), we deduce \((32)\).

**Remark.** The estimates of Proposition 2 are in accord with the stability result of Proposition 1, and they allow to prove a convergence of the numerical scheme.

### 3 Numerical Examples

#### 3.1 Computing strategy

The operator \(A\) defined in \((11)\) is well defined as a linear application \(X_N \rightarrow \mathbb{R}^{N+1}\), in the sense that we do not need additional point on the outside of \([0, L]\) to compute \(Au\). Thus \(A\) is represented by a penta-diagonal matrix of \((N+1) \times (N+1)\):

\[
Au = \begin{pmatrix}
\gamma_1 & \varepsilon_1 & \zeta_1 & 0 \\
\beta_2 & \gamma_2 & \varepsilon_2 & \zeta_2 \\
\alpha_3 & \beta_3 & \gamma_3 & \varepsilon_3 & \ldots & \\
& \ldots & \ldots & \ldots & \ldots & \\
& & & \ldots & \ldots & \zeta_{n-2} \\
& & & & \ldots & \beta_{n-1} & \gamma_{n-1} & \varepsilon_{n-1} \\
& & & & & \alpha_n & \beta_n & \gamma_n
\end{pmatrix}
\]

where \(\alpha_i, \beta_i, \gamma_i, \varepsilon_i\) and \(\zeta_i\), for \(i = 1, \ldots, n\) are coefficients easy to compute.

In the case of the implicit Euler approximation, the nonlinear system \((14)\) can be write as
\[
(I + A) u^{n+1} = u^n - \delta t F(u^{n+1})
\] (38)
where \(A = \delta t (\text{diag}(a_i) + A)\). The nonlinear system \((38)\) can be approximately solved using Newton method or a fixed point method. In both cases, we have in each iteration to solve a linear system with a define positive penta-diagonal matrix. Taking into account the structure of the matrix \((I + A)\) it is easy to apply a \(LU\) decomposition based on a simple modification of the Thomas algorithm for a penta-diagonal matrix.

In the case of the 2-stage Gauss-Legendre implicit Runge-Kutta method, the nonlinear system \((18)\) can be write as
\[
\begin{align*}
(I + a_{11} A) u^{n+1} + a_{12} A u^{n,2} &= u^n - \delta t a_{11} F(u^{n,1}) - \delta t a_{12} F(u^{n,2}) \\
&= u^n - \delta t a_{21} A u^{n,1} + (I + a_{22} A) u^{n,2} \\
&= u^n - \delta t a_{22} A F(u^{n,1}) - \delta t a_{22} F(u^{n,2})
\end{align*}
\]

where \(a_{ij}\) are defined in the Butcher table \((17)\), for \(i, j = 1, 2\). We can be rewrite this nonlinear system as an uncoupled system in its linear part as
\[
\begin{align*}
(12 I + 6 A + A^2) u^{n,1} &= (12 I + 2\sqrt{3} A) u^n - \delta t (3 I + A) F(u^{n,1}) \\
&= -\delta t (3 - 2\sqrt{3}) F(u^{n,2})
\end{align*}
\] (39)

\[
(12 I + 6 A + A^2) u^{n,2} = (12 I - 2\sqrt{3} A) u^n - \delta t (3 + 2\sqrt{3}) F(u^{n,1}) \\
&= -\delta t (3 + A) F(u^{n,2})
\] (40)

Using now Newton method or a fixed point method, we have in each iteration to solve a linear system with an 9-diagonal matrix \((12 I + 6 A + A^2)\). Taking into account the structure of the 9-diagonal matrix it is easy to apply again, a simple modification of the Thomas algorithm.

#### 3.2 Interaction between two solitons

An exact solution for the generalized KdV equation
\[
u_t + u_{xxx} + u^p u_x + u_x = 0, \quad (0, L) \times (0, + \infty),
\]
with \(x \in \mathbb{R}\) can be write as a traveling-wave solution (soliton) of the form
\[
u(x, t) = \frac{\alpha}{\cosh^{2/p} [\beta (x - (4\beta^2 + 1) t - x_0)]}
\]
where \(\alpha\) and \(x_0\) are arbitrary constants and \(\beta = \left[\frac{\alpha^p}{2(p+1)(p+2)}\right]^{1/2}\) (see for instance, Ablowitz and Segur [1]).

Obviously in a bounded domain there is not soliton, but the solution can be approximate by one if it is not close to a boundary. We choose \(L = 30\) and \(x_0 = 50\) during \(T = 100\) [sec] to avoid any numerical reflection due to the boundaries.
We consider an interaction of two solitons with $p = 4$. For that, we take the initial condition:

$$u(x, 0) = \frac{\alpha_1}{\cosh^{2/p}\left[\beta_1 p (x - (4 \beta_2 + 1) t - x_1)\right]} + \frac{\alpha_2}{\cosh^{2/p}\left[\beta_2 p (x - (4 \beta_2 + 1) t - x_2)\right]}$$

with $\alpha_1 = 1.0$, $\alpha_2 = 0.5$, $\beta_i = \left[\frac{\alpha_i^2}{2(p+1)(p+2)}\right]^{1/2}$, $i = 1, 2$, and $x_1 = 50.0$ and $x_2 = 80.0$. We make a simulation for $L = 300.0$; $T = 100$; $J = 10000$; $n = 100000$; $\delta t = T/n$; $\delta x = L/J$ (see Figure 1).

We remark that the norm $L^2$ of the initial condition of a soliton is given by

$$\|u_0\|_{L^2(0,L)} \approx 15^{1/4} \sqrt{\frac{\pi}{2}} \approx 2.4665092 > \sqrt{3/2}$$

Figura 1: Interaction between two solitons.

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Referências


