Abstract: The aim of this paper is to show the existence of solutions to a nonlinear problem arising from the isothermal solidification of a binary alloy in two dimensional domain. For this, a mathematical analysis of a time discretization scheme is considered. We establish a discrete maximum principle to obtain the a priori estimates. The convergence of the solutions of discrete scheme is proved and existence and regularity results for the original problem are derived.

1 Introduction

Phase separation observed during the solidification process of materials is an important and fascinating topic for researches. It is a difficult topic, though, and one of the principal obstacles in predicting solidification patterns is the difficulty of accurately calculating the processes occurring in the liquid and solid regions for the complex shapes of the liquid-solid interfaces that appear during the process. For the treatment of the such interfaces, the phase-field method has emerged as a powerful tool, among the large class of the methods that rely on treating a macroscopically sharp interface as a diffuse region. In this type of modeling, an extra variable, the so called phase-field \( \varphi(x,t) \), is introduced as an order parameter used to describe the phases of the material. The behavior of this variable is governed by a suitable equation that is then coupled to other equations obtained by the more usual balancing of mass, momentum, energy or solute, according the peculiar situation under investigation. Therefore, it is important to understand the mathematical problems resulting from this approach.

Following this line of research, the goal of this paper is to analyze a nonlinear parabolic system of highly nonlinear partial differential equations arising by using this kind of modeling in a isothermal solidification process of a binary alloy. More precisely, we consider a phase-field model motivated by the ones in [1, 2, 4, 11]. In the model to be analyzed, the state of the alloy is characterized by the relative concentration \( c \) (the proportion of solute in the solvent) and phase-field \( \varphi \). When \( \varphi = 0 \) the alloy is considered to be liquid, \( \varphi = 1 \) the alloy is solid. The region when \( 0 < \varphi < 1 \) corresponds to the solid-liquid transition region, which is sometimes called the mushy region.

Let \( \Omega \subset \mathbb{R}^2 \) be an open bounded domain with a \( C^2 \) boundary and \( Q = \Omega \times (0,T) \) the space-time cylinder with lateral surface \( S = \partial \Omega \times (0,T) \). Then, we consider the following nonlinear system (P):

\[
\begin{align*}
\partial_t \varphi - \xi^2 \Delta \varphi &= \varphi(\varphi-1)(1-2\varphi) - |\nabla \varphi| c \quad \text{in } Q, \\
\partial_t c - \text{div}(D_1(\varphi)\nabla c) &= \text{div}(D_2(c,\varphi)\nabla \varphi) \quad \text{in } Q, \\
\varphi &= 0, \quad c = 0 \quad \text{on } S, \\
\varphi(x,0) &= \varphi_0(x), \quad c(x,0) = c_0(x) \quad \text{in } \Omega.
\end{align*}
\]

Here, \( \xi \) is a positive constant associated to material properties; \( D_1(\cdot) \) and \( D_2(\cdot,\cdot) \) are diffusion coefficients of the solute in the matrix of the solvent, that is, the other material constituting the binary alloy; \( \varphi_0(\cdot) \) and \( c_0(\cdot) \) are the ini-
tial conditions respectively for the phase-field and solute concentration. For simplicity of exposition, we assumed homogeneous Dirichlet boundary conditions; with simple modifications of the arguments, similar analysis could be done for other kinds of boundary conditions; for instance, we could take homogeneous Neumann boundary conditions for some of all of the unknown; consider appropriate nonhomogeneous boundary conditions, and so on.

The previous phase-field equation was derived by Beckermann, Diepers, Steinbach in [1], but here it is presented in a form obtained there before a final simplification was carried out. Thus, in a certain sense the phase field equation given in (P) can be considered more accurate than the final form stated in [1]. The concentration equation is obtained by rather general exposition, we assumed homogeneous Dirichlet boundary conditions, and so on.

The next section is dedicated to fix the notation by M, and sometimes M_1, M_2, ..., constants depending only on known quantities.

2 Notation, assumptions and auxiliary results

We denote the Banach space \( W^{2,1}_q(Q) \) consists of functions \( u(x,t) \) in \( L^q(Q) \) whose generalized derivatives \( \partial_x u, \partial^2_x u, \partial_t u \) are \( L^q \)-integrable; here \( \partial^2_x \) denotes all the partial derivatives with respect to \( x_1, \ldots, x_n \) of order \( s \). Its norm is given by

\[
||u||^2_{q,Q} = ||u||^q_{q,Q} + ||\partial_x u||^q_{q,Q} + ||\partial^2_x u||^q_{q,Q} + ||\partial_t u||^q_{q,Q}.
\]

We refer to [5, 8], for instance, for more details of the previous spaces.

We will also need the following Gagliardo-Nirenberg type interpolation inequalities stated in the following lemmas (see Zheng [13, p. 3]):

**Lemma 2.1** Let \( \Omega \subset \mathbb{R}^d, d = 1, 2 \) or 3, is a bounded domain with a \( C^2 \)-boundary. Then, there exist a positive constant \( M \) such that

\[
||\nabla u||^2_{4,\Omega} \leq M ||\Delta u||^2_{2,\Omega} ||u||_{\infty,\Omega}, \quad (2.1)
\]

\( \forall u \in H^2_0(\Omega) \cap L^\infty(\Omega) \).

In lower dimensions, the following holds:

**Lemma 2.2** Let \( \Omega \subset \mathbb{R}^d, d = 1 \) or 2 is a bounded domain with a \( C^2 \)-boundary. Then, there exist a positive constant \( M \) such that

\[
||u||^2_{4,\Omega} \leq M ||u||^2_{2,\Omega} ||\nabla u||_{2,\Omega}, \quad (2.2)
\]

\( \forall u \in H^1_0(\Omega) \cap L^2(\Omega) \).

\[
||\nabla u||^2_{4,\Omega} \leq M ||\nabla u||^2_{2,\Omega} ||\Delta u||_{2,\Omega}, \quad (2.3)
\]

\( \forall u \in H^2(\Omega) \cap H^1(\Omega) \).

Throughout this paper we shall make the following assumptions:

**\( (H) \)**: \( \Omega \subset \mathbb{R}^2 \) is a bounded domain with \( C^2 \)-boundary;

\( D_1(\cdot) \subset C^0(\mathbb{R}) \) such that

\[
0 < \rho_0 \leq D_1(\cdot) \leq \rho_1, \quad \rho_0, \rho_1 \in \mathbb{R}^+;
\]

\( D_2(\cdot, \cdot) \subset C^0(\mathbb{R}^2) \) such that

\[
|D_2(\cdot, \cdot)| \leq \alpha_0, \quad \alpha_0 \in \mathbb{R}^+.
\]

In this paper we shall frequently make use of some auxiliary results, which we list below for further reference.

Frequently we shall use the discrete Gronwall Lemma:
Lemma 2.3 Let $\tau$, $M$, $u_n$, $v_n$ (for integers $n \geq 0$) be nonnegative numbers such that

$$u_n \leq M + \tau \sum_{k=0}^{n} v_k u_k \quad \text{for } n \geq 0.$$

Suppose that $\tau v_k < 1$. Then

$$u_n \leq M \exp \left( \tau \sum_{k=0}^{n} \frac{v_k}{1 - \tau v_k} \right) \quad \text{for } n \geq 0.$$ 

At various places we shall use the following relation

$$2 \int_{\Omega} (a - b) \, dx = ||a||_{L^2(\Omega)}^2 - ||b||_{L^2(\Omega)}^2 + ||a - b||_{L^2(\Omega)}^2. \quad (2.4)$$

Our next lemma provides a compactness criterion:

Lemma 2.4 Let $X$ and $Y$ be two (not necessarily reflexive) Banach spaces, such that $Y \subset X$, the injection being compact.

Assume that $G$ is a family of functions in $L^1(0, T; Y) \cap L^p(0, T; X)$ for some $T > 0$ and $p > 1$, such that

$$G \text{ is bounded in } L^1(0, T; Y) \text{ and } L^p(0, T; X); \quad (2.5)$$

$$\sup_{g \in G} \int_0^{T-a} ||g(a + s) - g(s)||_{L^p_X}^p \, ds \to 0 \quad (2.6)$$

as $a \to 0$, $a > 0$.

Then the family $G$ is relatively compact in $L^p(0, T; X)$.

2.1 Semi discretization

We consider a semi discretization of problem (P) with respect to the time variable as follows: we subdivide $[0, T]$ into $N$ subintervals $[t_{m-1}, t_m]$, $t_m = \tau m$, $\tau = T/N$, and for $m = 1, 2, \ldots, N$ we consider the following system of elliptic problems (PD):

$$\delta_t \varphi^m - \xi^2 \Delta \varphi^m = (\varphi^{m-1} - 1)(1 - 2\varphi^{m-1})\varphi^m - |\nabla \varphi^m|c^{m-1} \quad \text{a.e. in } \Omega,$$

$$\delta_t c^m - \text{div}(D_1(\varphi^m) \nabla c^m) = \text{div}(D_2(c^m, \varphi^m) \nabla \varphi^m) \quad \text{a.e. in } \Omega,$$

$$\varphi^0 = \varphi_0(x), \ c^0 = c_0(x), \ \delta_t \varphi^m = (\varphi^{m-1} - \varphi^m) / \tau, \ \delta_t c^m = (c^m - c^{m-1}) / \tau \text{ and } \varphi^m \text{ and } c^m, \ m = 1, \ldots, N, \text{ are expected to be approximations of } \varphi(x, t_m) \text{ and } c(x, t_m), \text{ respectively.}$$

We will prove the following existence result for discrete scheme (PD):

**Theorem 2.1** There exists an unique solution $(\varphi^m, c^m)$ in $H^2(\Omega) \times H^1_0(\Omega)$ of the approximate problem (PD).

With this result we may introduce the functions: for $t \in [t_{m-1}, t_m]$, and $1 \leq m \leq N$, we define $\bar{\varphi}_t(t) = \varphi^{m-1} + (t - t_{m-1})\delta_t \varphi^m$, $\bar{c}_t(t) = c^{m-1} + (t - t_{m-1})\delta_t c^m$, and corresponding step functions are given by $\varphi_t(t) = \varphi^m$ and $c_t(t) = c^m$.

The sequence of functions $(\bar{\varphi}_t, \bar{c}_t)$ is expected to converge as $\tau \to 0$, in suitable function spaces. The limit function $(\varphi, c)$ is expected to be a solution to the problem (P):

**Theorem 2.2** Assume that (H) holds. Suppose $(\varphi_0, c_0) \in H^1(\Omega) \cap L^2(\Omega)$ with $0 \leq \varphi_0(x) < 1$ for a.e. $x \in \Omega$. Then there exists a unique solution $(\varphi, c)$ satisfying $(\varphi, c) \in W_2^1(Q) \times (L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)))$ and $c_t \in L^2(0, T; H^{-1}(\Omega))$ such that $(\varphi(0), c(0)) = (\varphi_0, c_0)$ and

$$(c_t, v) + \int_{\Omega} D_1(\varphi) \nabla \varphi \nabla v \, dx = \int_{\Omega} D_2(\varphi, c) \nabla \varphi \nabla v \, dx,$$

for all $v \in H^1_0(\Omega)$ a.e. in $(0, T)$,

$$\varphi_t - \xi^2 \Delta \varphi = \varphi(1 - 2\varphi) - |\nabla \varphi|c \quad \text{in } Q,$$

$$\varphi = 0 \quad \text{on } S.$$

When $c_0$, $D_1$ and $D_2$ are smooth enough, we may have further regularity properties:

**Theorem 2.3** Assume that (H) holds. Suppose $c_0 \in H^1_0(\Omega)$, $D_1(\varphi)$ and $D_2(\varphi, c)$ are differentiable, with $\nabla D_1 \in L^\infty(\mathbb{R})$ and $\nabla D_2 \in L^\infty(\mathbb{R}^2)$. Then there exists an unique solution $(\varphi, c) \in (W_2^2(Q))^2$ of the problem (P).

3 Proof of Theorema 2.1

For a fixed $m$, assuming that $\varphi^{m-1}$ and $c^{m-1}$ are already known and we consider the following nonlinear system:

$$-\tau \xi^2 \Delta \varphi^m + \varphi^m = \tau (\varphi^{m-1} - 1)(1 - 2\varphi^{m-1})\varphi^m - \tau |\nabla \varphi^m|c^{m-1} + \varphi^m \quad \text{in } \Omega,$$

$$-\tau \text{div}(D_1(\varphi^m) \nabla c^m) + c^m = \tau \text{div}(D_2(c^m, \varphi^m) \nabla \varphi^m) + c^{m-1} \quad \text{in } \Omega,$$

$$\varphi^m = 0, \ c^m = 0 \quad \text{on } \partial \Omega.$$
4 Discrete maximum principle

Next, we establish a discrete maximum principle for the semi discrete problem:
\[
\delta_t \varphi_m^m - \xi^2 \Delta \varphi^m = (\varphi^{m-1} - 1)(1 - 2\varphi^{m-1})\varphi_m^m - \frac{1}{m} a.e. \text{ in } \Omega, \quad (4.7)
\]
where \( \varphi^0 = \varphi_0(x) \) such that \( 0 \leq \varphi_0(x) \leq 1 \) and 
\[
\delta_t \varphi^m = (\varphi^m - \varphi^{m-1})/\tau.
\]

Now, multiplying the equation (4.7) by \( 2\tau \varphi^m \), integrating over \( \Omega \) and using Green’s formula, we obtain
\[
2 \int_{\Omega} (\varphi^m - \varphi^{m-1}) \varphi_m^m \, dx + 2\tau \xi^2 \int_{\Omega} |\nabla \varphi^m|^2 \, dx = 0.
\]
Moreover, we can write
\[
2 \int_{\Omega} (\varphi^m - \varphi^{m-1}) \varphi_m^m \, dx + 2\tau \xi^2 \int_{\Omega} |\nabla \varphi^m|^2 \, dx = 0.
\]

Using the result interpolation (2.2) and Young’s inequality, we have
\[
||\varphi^m||_{2, \Omega}^2 - ||\varphi^{m-1}||_{2, \Omega}^2 + \frac{\tau \xi^2}{2} ||\nabla \varphi^m||_{2, \Omega}^2 \
\leq \tau M_1 ||\varphi^m||_{2, \Omega}^2 + \frac{\tau}{2\xi^2} ||\varphi^{m-1}||_{4, \Omega}^2 ||\varphi^m||_{2, \Omega}^2.
\]

Summing these relations for \( m = 1, 2, \ldots, r \), with \( 1 \leq r \leq N \), we find
\[
||\varphi^m||_{2, \Omega}^2 + 2\tau \xi^2 \sum_{m=1}^{r} ||\varphi^{m-1}||_{2, \Omega}^2 \leq \tau M_1 \sum_{m=1}^{r} ||\varphi^m||_{2, \Omega}^2 + \frac{\tau}{2\xi^2} ||\varphi^{m-1}||_{4, \Omega}^2 \sum_{m=1}^{r} ||\varphi^m||_{2, \Omega}^2.
\]

From discrete Gronwall Lemma 2.3
\[
M_2 ||\varphi^0||_{2, \Omega}^2 \exp \left( \frac{\tau}{2\xi^2} \sum_{m=1}^{r} ||\varphi^{m-1}||_{4, \Omega}^2 \right).
\]

Since \( \varphi^0 = 0 \), we conclude that \( ||\varphi^m||_{2, \Omega}^2 = 0 \).

Hence \( \varphi^m = 0 \) for all \( m \) and a.e. in \( \Omega \).

In the same way, \( \varphi^0 \leq 1 \) a.e in \( \Omega \) implies that \( \varphi^m \leq 1 \) for all \( m \) and a.e. in \( \Omega \); it is sufficient to multiply the equation (4.7) by \( 1 - \varphi^m \) and proceed similarity as before.

Then, we proved that
\[
0 \leq \varphi^m(x) \leq 1, \quad \forall m, \text{ a.e. in } \Omega. \quad (4.8)
\]

5 Proof of Theorem 2.2

Lemma 5.1 For \( \tau \) sufficiently small the following estimates are satisfied:
\[
||(c_t, \varphi_t)||_{L^\infty(0,T;H^1(\Omega))}^2 \leq M,
\]
\[
||(c_t, \varphi_t)||_{L^2(0,T;H^2(\Omega))}^2 \leq M,
\]
\[
||(c_t, \varphi_t)||_{L^\infty(0,T;L^2(\Omega))}^2 \leq M,
\]
\[
||(c_t, \varphi_t)||_{L^2(0,T;H^1(\Omega))}^2 \leq M,
\]
\[
||\varphi_t||_{2, Q}^2 \leq M.
\]

Moreover, \( ||(\varphi_t - \varphi_t, \varphi_t - c_t)||_{L^2(Q)}^2 \leq \tau M \).

The proof the Theorem 2.1 will be omitted.
5.1 Taking the limit

We deduce from Lemma 5.1 that \((\varphi_\tau, \varphi_\tau), (\bar{c}_\tau, c_\tau)\) are bounded (uniformly with respect to \(\tau\)) in \(W^{2,1}_2(Q) \times L^2(0,T; H^2(\Omega))\) and \((L^2(0,T; H^1_0(\Omega)))^2\), respectively. Moreover, \(W^{2,1}_2(Q)\) is compactly embedding into \(L^p(Q)\) with \(1 \leq p < \infty\) (see [9]). Hence, there exist \((\varphi, \varphi) \in W^{2,1}_2(Q) \times L^2(0,T; H^2(\Omega)), (\bar{c}, c) \in (L^2(0,T; H^1_0(\Omega)))^2\) and subsequences still denote \((\varphi_\tau, c_\tau), (\bar{c}_\tau, \bar{c}_\tau)\), such that as \(\tau \to 0\),

\[
\begin{align*}
\varphi_\tau & \to \varphi \text{ in } W^{2,1}_2(Q), \\
\varphi_\tau & \to \varphi \text{ in } L^2(0,T; H^2(\Omega)), \\
\bar{c}_\tau & \to \bar{c} \text{ in } L^2(0,T; H^1_0(\Omega)), \\
c_\tau & \to c \text{ in } L^2(0,T; H^1_0(\Omega)), \\
\bar{c}_\tau & \to \bar{c} \text{ in } L^\infty(0,T; L^2(\Omega)), \\
c_\tau & \to \bar{c} \text{ in } L^\infty(0,T; L^2(\Omega)).
\end{align*}
\]

Since \((\bar{\varphi}_\tau - \varphi_\tau, \bar{c}_\tau - c_\tau)\) converge to 0 in \((L^2(Q))^2\), we conclude that \((\varphi, c) = (\bar{\varphi}, \bar{c})\).

We infer from (5.9) that \(\varphi_\tau \to \varphi\) a.e. in \(Q\); this and \(0 \leq \varphi_\tau(x,t) \leq 1\) imply

\[
0 \leq \varphi(x,t) \leq 1 \text{ a.e. in } Q.
\]

Observe that from (5.9) and using the assumption (H), we obtain that

\[
D_1(\varphi_\tau) \to D_1(\varphi) \text{ a.e. in } Q.
\]

Now, for each \(\tau\) and any fixed \(2 \leq p < \infty\), define the functions \(g_\tau = |D_1(\varphi_\tau) - D_1(\varphi)|^p\).

By using again (H) and the previously proved almost everywhere convergences, we obtain \(|g_\tau| \leq 2^p p_1^p\) and \(g_\tau \to 0\) a.e. Then, by the Lebesgue’s dominated convergence theorem, we conclude that

\[
D_1(\varphi_\tau) \to D_1(\varphi) \text{ strongly in } L^p(Q), \quad (5.11)
\]

for any \(2 \leq p < \infty\).

Next, we need the estimate of fractional in time of \(\bar{c}_\tau\) to apply the compactness result given in Lemma 2.4.

We set \(Y = H^1_0(\Omega)\) and \(X = L^2(\Omega);\) the embedding of \(H^1_0(\Omega)\) in \(L^2(\Omega)\) is compact by Rellich’s theorem; we choose \(p = 2\) and \(\mathcal{G}\) is the family of functions \(\bar{c}_\tau\).

Now, using function \((\bar{\varphi}_\tau, \bar{c}_\tau)\), and corresponding step function \((\varphi_\tau, c_\tau)\) we rewrite (PD) in the form

\[
\begin{align*}
\frac{d}{dt} (\int_{\Omega} \bar{c}_\tau(t) \, dx) + & \int_{\Omega} D_1(\varphi_\tau(t)) \nabla c_\tau(t) \nabla v \, dx = \\
& - \int_{\Omega} D_2(\varphi_\tau(t), c_\tau(t)) \nabla \varphi_\tau(t) v \, dx,
\end{align*}
\]

for all \(v \in H^1_0(\Omega)\).

We then know from Lemma 5.1 the \(\bar{c}_\tau\) is bounded in \(L^2(0,T; H^1_0(\Omega))\). There remains to show (2.6). For this, we integrate (5.12) between \(t\) and \(t + a, t \in (0, T), a > 0\):

\[
\int_{\Omega} (\bar{c}_\tau(t + a) - \bar{c}_\tau(t)) v \, dx =
\]

\[
- \int_{t}^{t+a} \int_{\Omega} D_1(\varphi_\tau(s)) \nabla c_\tau(s) \nabla v \, dx \, ds
\]

\[
- \int_{t}^{t+a} \int_{\Omega} D_2(\varphi_\tau(s), c_\tau(s)) \nabla \varphi_\tau(s) \nabla v \, dx \, ds.
\]

We choose \(v = \bar{c}_\tau(t + a) - \bar{c}_\tau(t)\), and integrate these relations between 0 and \(T - a\). We find

\[
\int_{0}^{T-a} ||\bar{c}_\tau(t + a) - \bar{c}_\tau(t)||^2_{2,\Omega} = I_1 + I_2, \quad (5.13)
\]

where

\[
I_1 = - \int_{0}^{T-a} \int_{t}^{t+a} \int_{\Omega} D_1(\varphi_\tau(s)) J_1 \, dx \, ds \, dt,
\]

\[
I_2 = - \int_{0}^{T-a} \int_{t}^{t+a} \int_{\Omega} D_2(\varphi_\tau(s), c_\tau(s)) J_2 \, dx \, ds \, dt,
\]

with \(J_1 = \nabla c_\tau(s)(\nabla \bar{c}_\tau(t + a) - \nabla \bar{c}_\tau(t))\) and \(J_2 = \nabla \varphi_\tau(s)(\nabla \bar{c}_\tau(t + a) - \nabla \bar{c}_\tau(t))\).

Using the assumptions (H), Fubini’s theorem and Schwarz inequality, we get

\[
|I_1| \leq \rho_1 T^{1/2} a^{1/2} \left( \int_{0}^{T} ||\nabla c_\tau(s)||^2_{2,\Omega} \, ds \right)^{1/2},
\]

\[
\left( \int_{0}^{T-a} ||\nabla \bar{c}_\tau(t + a) - \nabla \bar{c}_\tau(t)||^2_{2,\Omega} \, dt \right)^{1/2},
\]

\[
|I_2| \leq \rho_2 T^{1/2} a^{1/2} \left( \int_{0}^{T} ||\nabla \varphi_\tau(s)||^2_{2,\Omega} \, ds \right)^{1/2},
\]

\[
\left( \int_{0}^{T-a} ||\nabla \bar{c}_\tau(t + a) - \nabla \bar{c}_\tau(t)||^2_{2,\Omega} \, dt \right)^{1/2}.
\]

From Lemma 5.1, we obtain

\[
|I_1| \leq M a^{1/2}, \quad |I_2| \leq M a^{1/2}. \quad (5.14)
\]

Combining (5.14) and (5.13), we obtain (2.6). Hence there exist subsequences still denote \(c_\tau\), such that as \(\tau \to 0\),

\[
c_\tau \to c \text{ in } L^2(Q). \quad (5.15)
\]
Applying arguments that are similar to the ones used to obtain the estimate (5.11), we can show the following convergence:

\[ D_2(\varphi, c) \to D_2(\varphi, c) \text{ strongly in } L^p(Q), \]

with \( 2 \leq p < \infty \).

On the other hand, for any \( \psi \in L^2(Q) \), we have that

\[
\int_Q ((3 \varphi^2 - 2 \varphi^3 - \varphi - 3 \varphi^2 - 2 \varphi^3 - \varphi)) \psi dx dt =
\]

\[
\int_Q d_\tau (\varphi - \varphi) \psi dx dt,
\]

where \( d_\tau = 3(\varphi_x + \varphi) - 2(\varphi_x^2 + \varphi_x + \varphi^2) - 1 \).

We observe that by using the fact that \( 0 \leq \varphi_r \leq 1 \) and \( 0 \leq \varphi \leq 1 \) we obtain \( ||d_\tau||_{\infty, Q} \leq M \). Consequently, \( \int_Q d_\tau (\varphi - \varphi) \psi dx dt \leq ||d||_{\infty, Q} ||\varphi - \varphi||_{2, Q} ||\psi||_{2, Q} \), and from (5.9), we conclude that

\[ \varphi(\varphi_r - 1)(1 - 2\varphi_r) \to \varphi(\varphi - 1)(1 - 2\varphi) \text{ in } L^2(Q). \]

Moreover, from (5.9) and (5.15), we conclude that

\[ |\nabla \varphi| c_r \to |\nabla \varphi| c \text{ in } L^2(Q). \]

With the previous convergence results, with standard arguments, it is easy to take the limit in the approximate problem (PD) and prove that \( \varphi \) and \( c \) is the required solution.

This completes the proof of Theorem 2.2. \Box

6 Proof of Theorem 2.3

To examine the regularity of \((\varphi, c)\), we assume that \( D_1(s) \in C^1(\mathbb{R}) \) and \( D_2(s, y) \in C^1(\mathbb{R}^2) \) such that

\[
|\partial_s D_1| \leq \rho_2, \quad |\partial_y D_2| \leq \alpha_1, \quad |\partial_s D_2| \leq \alpha_2,
\]

\[ (6.16) \]

where \( \rho_2, \alpha_1, \alpha_2 \in \mathbb{R}^+ \).

Now, we multiplying the second equation in (PD) by \(-2\tau \Delta c^m\), integrating over \( \Omega \) and using Green’s formula, we obtain

\[
2\int_\Omega (\nabla c^m - \nabla c^{m-1}) \nabla c^m \, dx +
\]

\[
2\tau \int_\Omega D_1(\varphi^m) |\Delta c^m|^2 \, dx =
\]

\[
2\tau \int_\Omega \partial_c D_1(\varphi^m) \nabla \varphi^m \nabla c^m \Delta c^m \, dx -
\]

\[
2\tau \int_\Omega \partial_c D_2(\varphi^m, \varphi^m) |\nabla \varphi^m|^2 \Delta c^m \, dx -
\]

\[
2\tau \int_\Omega \partial_\varphi D_2(\varphi^m, \varphi^m) \nabla \varphi^m \nabla c^m \Delta c^m \, dx -
\]

\[
-2\tau \int_\Omega D_2(\varphi^m, \varphi^m) \Delta \varphi^m \Delta c^m \, dx. \quad (6.17)
\]

Using the assumptions (H) and (6.16), the estimate (2.4), Hölder’s and Young’s inequalities, we obtain

\[
||\nabla c^m||^2_{2, \Omega} - ||\nabla c^{m-1}||^2_{2, \Omega} +
\]

\[
||\nabla c^m - \nabla c^{m-1}||^2_{2, \Omega} + \frac{\tau \rho_0}{8} ||\Delta c^m||^2_{2, \Omega} \leq
\]

\[
\tau M_1 ||\nabla \varphi^m||^2_{2, \Omega} + \tau M_2 ||\nabla c^m||^2_{2, \Omega} + \tau M_3 ||\Delta \varphi^m||^2_{2, \Omega}.
\]

From the interpolation result (2.1) and (4.8), we obtain

\[
||\nabla \varphi^m||^2_{2, \Omega} \leq M ||\Delta \varphi^m||_{2, \Omega}, \quad (6.18)
\]

and consequently, we have

\[
||\nabla c^m||^2_{2, \Omega} - ||\nabla c^{m-1}||^2_{2, \Omega} +
\]

\[
||\nabla c^m - \nabla c^{m-1}||^2_{2, \Omega} + \frac{\tau \rho_0}{8} ||\Delta c^m||^2_{2, \Omega} \leq
\]

\[
\tau M_4 ||\Delta \varphi^m||_{2, \Omega} ||\nabla c^m||_{2, \Omega} ||\Delta c^m||_{2, \Omega} +
\]

\[
\tau M_5 ||\Delta \varphi^m||^2_{2, \Omega}.
\]

Applying Young’s inequality, we get

\[
||\nabla c^m||^2_{2, \Omega} - ||\nabla c^{m-1}||^2_{2, \Omega} +
\]

\[
||\nabla c^m - \nabla c^{m-1}||^2_{2, \Omega} + \frac{\tau \rho_0}{16} ||\Delta c^m||^2_{2, \Omega} \leq
\]

\[
\tau M_6 ||\Delta \varphi^m||^2_{2, \Omega} ||\nabla c^m||^2_{2, \Omega} +
\]

\[
\tau M_5 ||\Delta \varphi^m||^2_{2, \Omega}.
\]

Summing these relations for \( m = 1, 2, ..., r \), with \( 1 \leq r \leq N \), we find

\[
||\nabla \varphi^r||^2_{2, \Omega} + \sum_{m=1}^r ||\nabla c^m - \nabla c^{m-1}||^2_{2, \Omega} +
\]

\[
\tau \sum_{m=1}^r ||\Delta c^m||^2_{2, \Omega} \leq M_7 (||\nabla \varphi_0||^2_{2, \Omega} +
\]

\[
\tau \sum_{m=1}^r ||\Delta \varphi^{m-1}||^2_{2, \Omega} ||\nabla c^m||^2_{2, \Omega} + \tau \sum_{m=1}^r ||\Delta \varphi^m||^2_{2, \Omega}).
\]

From Lemma 5.1 and discrete Gronwall Lemma 2.3, we obtain

\[
\max_{1 \leq r \leq N} ||\nabla \varphi^r||_{2, \Omega} + \tau \sum_{m=1}^r ||\Delta c^m||^2_{2, \Omega} +
\]

\[
\sum_{m=1}^r ||\nabla c^m - \nabla c^{m-1}||^2_{2, \Omega} \leq
\]

\[
\tau \sum_{m=1}^r ||\Delta \varphi^{m-1}||^2_{2, \Omega} ||\nabla c^m||^2_{2, \Omega} + \tau \sum_{m=1}^r ||\Delta \varphi^m||^2_{2, \Omega}).
\]
\[ M_8 \left( 1 + ||\varphi_0||_{\mathcal{H}^1(\Omega)}^2 + ||e^0||_{H^1(\Omega)}^2 \right). \] (6.19)

Now, we multiplying the concentration equation in (PD) by \( \delta_t e^m \), integrating over \( \Omega \) and using Green’s formula, we obtain

\[
\begin{align*}
||\delta_t e^m||^2_{2,\Omega} &= \int_\Omega \partial_x D_1(\varphi^m) \nabla \varphi^m \nabla e^m \delta_t e^m \, dx + \\
&\quad \int_\Omega D_1(\varphi^m) \Delta e^m \delta_t e^m \, dx - \\
&\quad \int_\Omega \partial_x D_2(e^m, \varphi^m) |\nabla \varphi^m|^2 \delta_t e^m \, dx - \\
&\quad \int_\Omega \partial_c D_2(e^m, \varphi^m) \nabla \varphi^m \nabla \delta_t e^m \, dx - \\
&\quad \int_\Omega D_2(e^m, \varphi^m) \Delta \varphi^m \delta_t e^m \, dx.
\end{align*}
\]

Using the assumptions (H) and (6.16), Hölder’s and Young’s inequalities, we obtain

\[
\begin{align*}
||\delta_t e^m||^2_{2,\Omega} &\leq M_9 \left( ||\nabla \varphi^m||^2_{4,\Omega} ||\nabla e^m||^2_{2,\Omega} + \\
&\quad ||\nabla \varphi^m||^4_{4,\Omega} + ||\Delta \varphi^m||^2_{2,\Omega} + ||\Delta e^m||^2_{2,\Omega} \right).
\end{align*}
\]

From (2.3) and (6.18), we get

\[
\begin{align*}
||\delta_t e^m||^2_{2,\Omega} &\leq \\
&\quad M_{10} \left( ||\Delta \varphi^m||^2_{2,\Omega} ||\nabla e^m||^2_{2,\Omega} + \\
&\quad ||\Delta \varphi^m||^2_{2,\Omega} + ||\Delta e^m||^2_{2,\Omega} \right).
\end{align*}
\]

Applying Young’s inequality, we have

\[
\tau ||\delta_t e^m||^2_{2,\Omega} \leq \\
\quad \tau M_{11} \left( ||\Delta \varphi^m||^2_{2,\Omega} ||\nabla e^m||^2_{2,\Omega} + \\
&\quad ||\Delta \varphi^m||^2_{2,\Omega} + ||\Delta e^m||^2_{2,\Omega} \right).
\]

Summing these relations for \( m = 1, 2, \ldots, N \), we find

\[
\begin{align*}
\tau \sum_{m=1}^N ||\delta_t e^m||^2_{2,\Omega} &\leq M_{11} \left( \max_{1 \leq m \leq N} ||\nabla e^m||^2_{2,\Omega} + \\
&\quad \left( \tau \sum_{m=1}^N ||\Delta \varphi^m||^2_{2,\Omega} + \right) + \\
&\quad \tau \sum_{m=1}^N \sum_{m=1}^N ||\Delta \varphi^m||^2_{2,\Omega} \right).
\end{align*}
\]

From Lemma 5.1 and (6.19), we obtain

\[
\tau \sum_{m=1}^N ||\delta_t e^m||^2_{2,\Omega} \leq M_{12} \left( ||\varphi_0||^2_{H^1(\Omega)} + ||c_0||^2_{H^1(\Omega)} \right),
\]

where the constant \( M_{12} \) depends on \( T, \xi^2, \xi^4, \rho_1, \alpha, \Omega, ||\varphi_0||_{2,\Omega} \) and \( ||c_0||_{2,\Omega} \) with \( l = 1, 2 \).

This completes the proof of Theorem 2.3. \( \blacksquare \)

### References


