Asymptotics for Jacobi-Sobolev orthogonal polynomials associated with non-coherent pairs of measures

Eliana X.L. de Andrade, Cleonice F. Bracciali, A. Sri Ranga
Depto de Ciências de Computação e Estatística, IBILCE, UNESP, 15054-000, São José do Rio Preto, SP
E-mail: eliana@ibilce.unesp.br, cleonice@ibilce.unesp.br, ranga@ibilce.unesp.br

Abstract: Inner products of the type \( \langle f, g \rangle_S = \langle f, g \rangle_{\psi_0} + \langle f', g' \rangle_{\psi_1} \), where one of the measures \( \psi_0 \) or \( \psi_1 \) is the measure associated with the Jacobi polynomials, are usually referred to as Jacobi-Sobolev inner products. This paper deals with some asymptotic relations for the orthogonal polynomials with respect to a class of Jacobi-Sobolev inner products. The inner products are such that the associated pairs of measures \((\psi_0, \psi_1)\) are not within the concept of coherent pairs of measures.

Keywords: Orthogonal polynomials, Sobolev orthogonal polynomials, Asymptotics

1 Introduction

Consider the inner product defined by
\[
\langle f, g \rangle_{\psi(\alpha,\beta)} = \int_{-1}^{1} f(x)g(x)d\psi(x) = \int_{-1}^{1} f(x)g(x)(1-x)^\alpha(1+x)^\beta dx,
\]
with \( \alpha, \beta > -1 \). Many properties of the sequence of orthogonal polynomials \( \{P_n(\alpha,\beta)\}_n \) with respect to this inner product, known as the Jacobi polynomials, are well known. Here, we assume \( \{P_n(\alpha,\beta)\}_n \) to be a sequence of monic polynomials.

For example, it is known that
\[
P_{n+1}(\alpha,\beta) = (x - \beta_{n+1})P_n(\alpha,\beta) - \alpha_{n+1}P_1(\alpha,\beta), \quad n \geq 1,
\]
where \( \beta_{n+1} \), \( n \geq 0 \), are real numbers and
\[
\alpha_{n+1} = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)(2n+\alpha+\beta+1)} > 0, \quad n \geq 1.
\]
Moreover, the \( n \) zeros of \( P_n(\alpha,\beta) \) are simple and lie inside \((-1, 1)\). For more details about these polynomials see, for example, [4, 11].

Assume \( |\kappa| \geq 1, \kappa_2 \geq 0, \kappa_3 \geq 0 \) and \( \kappa_1 \geq -\frac{|\kappa|}{1 + |\kappa|} \kappa_2 \). We consider the class of Sobolev inner products \( \langle f, g \rangle_S \) defined as follows
\[
\langle f, g \rangle_S = \langle f, g \rangle_{\psi(\alpha,\beta)} + \kappa_1 \langle f', g' \rangle_{\psi(\alpha+1,\beta+1)} + \kappa_2 \langle f', g' \rangle_{\psi(\alpha,\beta,\kappa_3)},
\]
Here, the measure \( \psi(\alpha,\beta,\kappa_3) \) is such that
\[
\langle f, g \rangle_{\psi(\alpha,\beta,\kappa_3)} = \int_{-1}^{1} f(x)g(x)d\psi(x) + \kappa_3 \left[ f(\kappa)g(\kappa) \right],
\]
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with $d\psi^{(\alpha,\beta,\kappa)}(z) = \frac{\kappa}{\kappa - z} d\psi^{(\alpha+1,\beta+1)}(z)$. With the restrictions given before over $\kappa, \kappa_1, \kappa_2$ and $\kappa_3$ the inner product given in (2) is positive definite.

If $\kappa_1 \neq 0$, the pair of measures $\{d\psi^{(\alpha,\beta)}, \kappa_1 d\psi^{(\alpha+1,\beta+1)} + \kappa_2 d\psi^{(\alpha,\beta,\kappa_3)}\}$ does not form a coherent pair according to Meijer’s classification given in [6]. The concept of coherent pair of measures was introduced by Iserles et al. [5]. In Andrade, Bracciali, Mello, Pérez [1] the authors studied the behaviour of the zeros of Jacobi-Sobolev orthogonal polynomials for $\kappa_1 \neq 0$. In [2] one can find asymptotic properties for Gegenbauer–Sobolev orthogonal polynomials associated with non-coherent pairs of measures.

For $\kappa_1 = 0$ see ([7]) and ([9]) for asymptotic properties of Sobolev orthogonal polynomials for coherent pairs of Jacobi type.

If we denote the monic orthogonal polynomials with respect to the inner product given in (3) by $P_n^{(\alpha,\beta,\kappa_2)}$, the following results are also known (see [3]).

\[
P_n^{(\alpha,\beta,\kappa_2)}(x) = P_n^{(\alpha+1,\beta+1)}(x) + d_{n-1}(\kappa) P_{n-1}^{(\alpha+1,\beta+1)}(x), \quad n \geq 1,
\]

where $d_{n-1}(\kappa) = -\rho_n^{(\alpha,\beta,\kappa_2)}/[\kappa \rho_{n-1}^{(\alpha+1,\beta+1)}]$. Here, $\rho_n^{(\alpha,\beta,\kappa_2)} = \langle P_n^{(\alpha,\beta,\kappa_2)}, P_n^{(\alpha,\beta,\kappa_2)} \rangle_{\psi^{(\alpha,\beta,\kappa_2)}}$ and $\rho_n^{(\alpha+1,\beta+1)} = \langle P_n^{(\alpha+1,\beta+1)}, P_n^{(\alpha+1,\beta+1)} \rangle_{\psi^{(\alpha+1,\beta+1)}}$.

It was shown in [3] that the polynomials $S_n$ satisfy $S_0(x) = P_0^{(\alpha,\beta)}(x) = 1$, $S_1(x) = P_1^{(\alpha,\beta)}(x) = x$ and

\[
S_{n+1}(x) + a_n S_n(x) = P_{n+1}^{(\alpha,\beta)}(x) + b_n P_n^{(\alpha,\beta)}(x), \quad n \geq 1,
\]

where $b_n = b_n(\kappa) = d_{n-1}(\kappa)(n+1)/n$ and for $n \geq 1$,

\[
a_n = a_n(\kappa, \kappa_1, \kappa_2) = \frac{\rho_n^{(\alpha,\beta)} + \kappa_1 \rho_{n-1}^{(\alpha+1,\beta+1)}}{\rho_n^{(S)}} b_n(g),
\]

where $\rho_n^{(S)} = \langle S_n, S_n \rangle_S$. The coefficients $a_n, n \geq 2$, can be recursively generated by

\[
a_n = \frac{\nu_n(\kappa_1) + \alpha_n^{(\alpha+1,\beta+1)}}{\nu_n(\kappa_1) + \alpha_n^{(\alpha+1,\beta+1)} + b_n \{ - n(n-1)\kappa_2 \kappa + \nu_{n-1}(\kappa_1)[b_{n-1} - a_{n-1}] \}} b_n,
\]

with $a_1 = \frac{\nu_1(\kappa_1)}{\nu_1(\kappa_1) + \kappa_2 \rho_0^{(\alpha,\beta,\kappa_2)}/\rho_0^{(\alpha+1,\beta+1)}} b_1$ and $\nu_n(\kappa_1) = n^2 \kappa_1 + n/(n + \alpha + \beta + 1), \quad n \geq 1$.

Our objective in this paper is to consider asymptotic results associated with the orthogonal polynomials $S_n$ with respect to the inner product $\langle f, g \rangle_S$ in (2) when $|\kappa| \geq 1$, $\kappa_1 > 0$, $\kappa_2 > 0$ and $\kappa_3 \geq 0$.

### 2 Some preliminary asymptotic results

Using results given in [4] or [11] one can easily verify that as $n \to \infty$,

\[
\frac{\rho_n^{(\alpha,\beta)}}{\rho_{n+1}^{(\alpha,\beta)}} \to 4 \quad \text{and} \quad \frac{\rho_n^{(\alpha,\beta)}}{\rho_n^{(\alpha+1,\beta+1)}} \to 1.
\]

We now give the asymptotic behaviour of the coefficients $d_n(\kappa)$ as well as of the rational functions such as $P_n^{(\alpha,\beta,\kappa_2)}(x)/P_n^{(\alpha+1,\beta+1)}(x)$ and $P_n^{(\alpha,\beta,\kappa_3)}(x)/P_n^{(\alpha,\beta)}(x)$, in terms of the complex function $\Phi$ defined by

\[
\Phi(z) = z + \sqrt{z^2 - 1}, \quad \text{for } z \in \mathbb{C} \setminus [-1, 1].
\]

The square root in $\Phi$ is such that $\sqrt{z^2 - 1} > 0$ when $z > 1$ and $\sqrt{z^2 - 1} < 0$ when $z < -1$. Here, $\mathbb{C}$ denotes the complex plane and $\overline{\mathbb{C}}$ denotes the extended complex plane.
Fist we will need some previous results.

From results given in Nevai [8] (see also Pan [9, Lemma 3.1]) we obtain the well known ratio asymptotics for the Jacobi polynomials

\[
\lim_{n \to \infty} \frac{P_{n+1}^{(\alpha, \beta)}(x)}{P_n^{(\alpha, \beta)}(x)} = \frac{\Phi(x)}{2},
\]

(9)

\[
\lim_{n \to \infty} \frac{P_{n}^{(\alpha+1, \beta+1)}(x)}{P_n^{(\alpha, \beta)}(x)} = \frac{\Phi(x)}{2\sqrt{x^2 - 1}} = \frac{\Phi'(x)}{2},
\]

(10)

uniformly on compact subsets of \( \mathbb{C} \setminus [-1, 1] \).

Now we state the following lemma given in Pan [9, Lemmas 4.3, 4.4 and 4.5].

**Lemma 2.1** Uniformly on compact subsets of \( \mathbb{C} \setminus [-1, 1] \),

\[
\lim_{n \to \infty} \frac{P_{n}^{(\alpha, \beta, \kappa, \kappa^3)}(x)}{P_n^{(\alpha, \beta)}(x)} = \begin{cases} 
1 - \frac{\Phi(x) - \Phi(\kappa)}{x - \kappa} \frac{\sqrt{\kappa^2 - 1}}{\Phi(x)}, & \text{if } \kappa_3 \geq 0, \\
1, & \text{if } \kappa_3 = 0,
\end{cases}
\]

(11)

\[
\lim_{n \to \infty} \frac{P_{n}^{(\alpha+1, \beta+1)}(x)}{P_n^{(\alpha, \beta)}(x)} = \frac{\Phi(x) - \Phi(\kappa)}{2(x - \kappa)},
\]

(12)

\[
\lim_{n \to \infty} \frac{P_{n}^{(\alpha, \beta, \kappa)}(x)}{P_n^{(\alpha, \beta)}(x)} = \frac{x - \kappa}{\Phi(x) - \Phi(\kappa)} \Phi'(x).
\]

(13)

We will also need the two results bellow.

**Lemma 2.2 (Pan [9], Thm. 4.6)** The following limit holds

\[
\lim_{n \to \infty} \frac{P_{n}^{(\alpha, \beta, \kappa, \kappa^3)}(x)}{P_n^{(\alpha, \beta)}(x)} = \begin{cases} 
\frac{x - \kappa}{\Phi(x) - \Phi(\kappa)} \Phi'(x) - \frac{\sqrt{\kappa^2 - 1}}{\sqrt{x^2 - 1}}, & \text{if } \kappa_3 \geq 0, \\
\frac{x - \kappa}{\Phi(x) - \Phi(\kappa)} \Phi'(x), & \text{if } \kappa_3 = 0,
\end{cases}
\]

(14)

locally uniformly in \( \mathbb{C} \setminus [-1, 1] \).

**Lemma 2.3 (Pan [9], Thm. 4.7)** For the coefficients \( d_n(\kappa) \) in (4) and also for the coefficients \( b_n(\kappa) \) in (5) the following holds

\[
\lim_{n \to \infty} d_n(\kappa) = \lim_{n \to \infty} b_n(\kappa) = b(\kappa),
\]

(15)

where \( 2b(\kappa) = \begin{cases} 
-\Phi(\kappa), & \text{if } \kappa_3 > 0, \\
-\Phi(\kappa)^{-1}, & \text{if } \kappa_3 = 0.
\end{cases} \)

From (4) and previous lemma we obtain

**Corollary 2.1**

\[
\lim_{n \to \infty} \frac{P_{n}^{(\alpha, \beta, \kappa, \kappa^3)}(x)}{P_n^{(\alpha+1, \beta+1)}} = \begin{cases} 
\frac{\kappa \Phi(\kappa)}{2}, & \text{if } \kappa_3 > 0, \\
\frac{\kappa}{2\Phi(\kappa)}, & \text{if } \kappa_3 = 0.
\end{cases}
\]
3 Asymptotic properties

The following theorem gives information for the norm $\|S_n\|_S = [\rho_n^S]^{1/2}$.

**Theorem 3.1**

\[
\gamma_n^{(\alpha,\beta,\kappa_1)} + n^2 \kappa_2 \rho_{n-1}^{(\lambda,\xi,\kappa_1,\kappa_2)} \leq \rho_n^S \leq \gamma_n^{(\alpha,\beta,\kappa_1)} + n^2 \kappa_2 \rho_{n-1}^{(\alpha,\beta,\kappa,\kappa_3)} + b_{n-1}(q) \gamma_{n-1}^{(\alpha,\beta,\kappa_1)},
\]

where $\gamma_n^{(\alpha,\beta,\kappa_1)} = \rho_n^{(\alpha,\beta)} + \kappa_1 \rho_{n-1}^{(\alpha+1,\beta+1)}$, $n \geq 1$.

**Proof:** Since the monic orthogonal polynomial of degree $n$ with respect to any inner product has the smallest norm among all the monic polynomials of degree $n$, we have

\[
\rho_n^S = \langle S_n, S_n \rangle_S = \langle S_n, S_n \rangle_{\psi(\alpha,\beta)} + \kappa_1 \langle S'_n, S'_n \rangle_{\psi(\alpha+1,\beta+1)} + \kappa_2 \langle S'_n, S'_n \rangle_{\psi(\alpha,\beta,\kappa,\kappa_3)}
\]

\[
\geq \rho_n^{(\alpha,\beta)} + n^2 \kappa_1 \rho_{n-1}^{(\alpha+1,\beta+1)} + n^2 \kappa_2 \rho_{n-1}^{(\alpha,\beta,\kappa,\kappa_3)}.
\]

To prove the right side inequality, we use

\[
\rho_n^S = \langle S_n, S_n \rangle_S \leq \langle P_n^{(\alpha,\beta)} + b_{n-1}(\kappa) P_n^{(\alpha,\beta)} \rangle_{\rho_{n-1}} + b_{n-1}(\kappa) P_{n-1}^{(\alpha,\beta)} \rangle_S.
\]

We can now prove the following result.

**Theorem 3.2** *The coefficients $a_n = a_n(\kappa, \kappa_1, \kappa_2)$ in (5) satisfy*

\[
\lim_{n \to \infty} a_n(\kappa, \kappa_1, \kappa_2) = a(\kappa, \kappa_1, \kappa_2) = -\frac{1}{2\Phi(\tilde{\kappa})}, \quad \text{where } \tilde{\kappa} = \frac{\kappa(\kappa_1 + \kappa_2)}{\kappa_1}.
\]

**Proof:** From previous theorem we obtain

\[
b_n [b_n - a_n] \geq 0 \quad \text{and} \quad |a_n| \leq \frac{\gamma_n^{(\alpha,\beta,\kappa_1)}}{\gamma_n^{(\alpha,\beta,\kappa_1)} + n^2 \kappa_2 \rho_{n-1}^{(\alpha,\beta,\kappa,\kappa_3)}} |b_n|.
\]

If $a = \lim_{n \to \infty} a_n$ exists, then from the inequality in Theorem 3.1, Eqs. (8) and (??) and Lemma 2.3,

\[
0 \leq |a| \leq \frac{\kappa_1}{\kappa_1 - 4\kappa_2 \gamma b} |b|.
\]

From (7), we obtain

\[
a_n = \frac{\delta_n a_n^{(\alpha+1,\beta+1)}}{\delta_n a_n^{(\alpha+1,\beta+1)} + b_{n-1} \frac{n-1}{n} \left[-\kappa_2 \kappa + \delta_n - \frac{n-1}{n} (b_{n-1} - a_{n-1})\right]} b_n,
\]

where $\delta_n = \nu_n(\kappa_1)/n^2$. From (1) note that $a_n^{(\alpha+1,\beta+1)} \to 1/4$ and $\delta_n \to \kappa_1$ as $n \to \infty$. Thus, if $a = \lim_{n \to \infty} a_n$ exists, then from (18),

\[
a^2 - \left[ \frac{1}{4b(\kappa)} + b(\kappa) - \frac{\kappa_2 \kappa}{\kappa_1} \right] a + \frac{1}{4} = 0.
\]

Both possibilities for $b(\kappa)$, given in Lemma 2.3, lead to

\[
a^2 + \kappa \left( 1 + \frac{\kappa_2}{\kappa_1} \right) a + \frac{1}{4} = 0.
\]
Hence, choosing the solution that satisfies the restriction given by (17), we obtain

$$a = -\frac{1}{2} \left[ \Phi \left( \frac{\kappa(\kappa_1 + \kappa_2)}{\kappa_1} \right) \right]^{-1}.$$  

We now confirm that \( \lim_{n \to \infty} a_n = a \) as given before. From (18), we obtain

$$\left| a_n - a \right| \leq \frac{\delta_n a_n^{(\alpha+1,\beta+1)}(b_n - a) + \kappa_2 \kappa - \delta_n - 1 b_{n-1} + \delta_n - 1}{\delta_n a_n^{(\alpha+1,\beta+1)} + b_{n-1} \frac{n-1}{n} \left[ -\kappa_2 \kappa + \delta_n - 1 \frac{n-1}{n} (b_{n-1} - a_{n-1}) \right]}$$

$$+ \frac{\delta_n - 1 b_{n-1} a \left( a_{n-1} - a \right)}{\delta_n a_n^{(\alpha+1,\beta+1)} + b_{n-1} \frac{n-1}{n} \left[ -\kappa_2 \kappa + \delta_n - 1 \frac{n-1}{n} (b_{n-1} - a_{n-1}) \right]}$$

$$\leq \frac{\delta_n a_n^{(\alpha+1,\beta+1)}(b_n - a) + \kappa_2 \kappa - \delta_n - 1 b_{n-1} + \delta_n - 1}{\delta_n a_n^{(\alpha+1,\beta+1)} - \frac{n-1}{n} \kappa_2 \kappa b_{n-1}}$$

$$+ \frac{\delta_n - 1 b_{n-1} a \left( a_{n-1} - a \right)}{\delta_n a_n^{(\alpha+1,\beta+1)} - \frac{n-1}{n} \kappa_2 \kappa b_{n-1}}.$$  

The latter part above is a consequence of \( b_n \geq 0 \). From (19),

$$\lim_{n \to \infty} \left[ \delta_n a_n^{(\alpha+1,\beta+1)}(b_n - a) + \kappa_2 \kappa - \delta_n - 1 b_{n-1} + \delta_n - 1 \right] \frac{n-1}{n} b_{n-1} a$$

$$= \kappa_1 b \left[ a^2 - \left( \frac{1}{4b} - \frac{\kappa_2 \kappa}{\kappa_1} \right) b + a + \frac{1}{4} \right] = 0.$$  

Therefore,

$$\limsup |a_n - a| \leq \frac{\kappa_1}{\kappa_1 - 4b\kappa_2} \left| 4ba \right| \limsup |a_{n-1} - a|.$$  

Thus, the convergence of \( a_n \) is established if we prove that \( \left| \frac{\kappa_1}{\kappa_1 - 4b\kappa_2} \right| |4ba| < 1 \). Clearly,

$$\left| \frac{\kappa_1}{\kappa_1 - 4b\kappa_2} \right| < 1 \text{ with the assumptions } \kappa_1 > 0.$$  

Now,

$$|4ab| = \begin{cases} \Phi(\kappa) & \text{if } \kappa_3 > 0, \\ \Phi(\kappa) & \text{if } \kappa_3 = 0. \end{cases}$$  

Since \( |\kappa| \geq 1 \) and \( |\Phi(\kappa)| \geq |\Phi(\kappa)| \geq 1 \) then \( |4ab| < 1 \). Thus, the theorem is proved. \hfill \Box

Using that \( a_n = b_n(\kappa) [\rho_n^{(a,\beta)} + n^2 \kappa_1 \rho_{n-1}^{(a+1,\beta+1)}] / \rho_n^S \) and \( b_n(\kappa) = -[(n-1)\rho_n^{(a,\beta,\kappa,\kappa_3)}] / [n \kappa \rho_{n-1}^{(a+1,\beta+1)}] \), together with the results in (8), Lemma 2.3 and Theorem 3.2, we obtain the following limit

$$\lim_{n \to \infty} \frac{n^2 \rho_n^{(a,\beta,\kappa,\kappa_3)}}{\rho_n^S} = \frac{\kappa}{2\kappa_1 \Phi(\kappa)}.$$  

**Theorem 3.3** The Sobolev orthogonal polynomials \( S_n(x) \) satisfy

$$\lim_{n \to \infty} \frac{S_n(x)}{P_n^{(a,\beta)}(x)} = \begin{cases} \Phi(x) - \Phi(\kappa) & \text{if } \kappa_3 > 0, \\ \Phi(x) - \frac{1}{\Phi(\kappa)} & \text{if } \kappa_3 = 0. \end{cases}$$  

$$\lim_{n \to \infty} \frac{S_{n+1}(x)}{S_n(x)} = \frac{\Phi(x)}{2},$$

uniformly on compact subsets of \( \overline{\mathbb{C}} \setminus [-1, 1] \).
**Proof:** From the recurrence relation (5), we can write
\[
 f_{n+1}(x) = 1 + g_n(x) + h_n(x)f_n(x),
\]  
(22)
with the analytic functions on \( \mathbb{C} \setminus [-1,1] \)
\[
f_n(x) = \frac{S_n(x)}{P_n^{(\alpha,\beta)}(x)}, \quad g_n(x) = b_n(\kappa)\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(x)} \quad \text{and} \quad h_n(x) = -a_n\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(x)}.
\]
Then equations (15), (9) and (16) give
\[
\lim_{n \to \infty} g_n(x) = g(x) = \frac{2b(\kappa)}{\Phi(x)} \quad \text{and} \quad \lim_{n \to \infty} h_n(x) = h(x) = -\frac{2a}{\Phi(x)},
\]  
(23)
uniformly on compact subsets of \( \mathbb{C} \setminus [-1,1] \).

Note that \( |h(x)| < 1 \) and that \( |g(x)| \) is also bounded for all \( x \in \mathbb{C} \setminus [-1,1] \). Hence there exist positive constants \( A < 1 \) and \( B \) such that for all \( n \geq N \)
\[
|h_n(x)| \leq A < 1 \quad \text{and} \quad |g_n(x)| \leq B \quad \text{if} \quad x \in \mathbb{C} \setminus [-1,1].
\]
Hence from (22),
\[
|f_{N+1}(x)| \leq 1 + B + A|f_N(x)|, \quad |f_{N+2}(x)| \leq 1 + B + A(1 + B) + A^2|f_N(x)|, \quad \vdots \quad |f_{N+i}(x)| \leq \frac{(1 + B)(1 - A^i)}{1 - A} + A^i|f_N(x)| < \frac{1 + B}{1 - A} + |f_N(x)|.
\]
Therefore, \( f_n \) is uniformly bounded on compact subsets of \( \mathbb{C} \setminus [-1,1] \).

We now show that \( \{f_n\} \) converges uniformly on compact subsets of \( \mathbb{C} \setminus [-1,1] \). If the limit \( f(x) \) exists, then from (22) it satisfies
\[
f(x) = 1 + g(x) + h(x)f(x).
\]  
(24)
Thus, from (23),
\[
f(x) = \frac{\Phi(x) + 2b}{\Phi(x) + 2a}.
\]  
(25)
From (22) and (24),
\[
|f_{n+1}(x) - f(x)| \leq |g_n(x) - g(x)| + |h_n(x) - h(x)||f_n(x)| + |h(x)||f_n(x) - f(x)|.
\]
Since \( f_n \) is bounded for all \( x \in \mathbb{C} \setminus [-1,1] \),
\[
\limsup|f_{n+1}(x) - f(x)| \leq A \limsup|f_n(x) - f(x)|.
\]
Consequently, the convergence of \( f_n(x) \) to \( f(x) \) follows from \( 0 < A < 1 \).

Using the results of Lemma 2.3 and Theorem 3.2, from (25) we obtain the limit in (20).

Since
\[
\frac{S_{n+1}}{S_n} = \frac{S_{n+1}(x)}{P_{n+1}^{(\alpha,\beta)}(x)} \frac{P_{n+1}^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(x)} \frac{P_n^{(\alpha,\beta)}(x)}{P_{n+1}^{(\alpha,\beta)}(x)} = \frac{S_n(x)}{P_n^{(\alpha,\beta)}(x)} \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(x)},
\]
from (20) and (9), the limit result in (21) immediately follows. \( \Box \)

Now, as we can write
\[
\frac{S_n(x)}{P_n^{(\alpha,\beta,\kappa,\kappa_3)}(x)} = \frac{S_n(x)}{P_n^{(\alpha,\beta)}(x)} \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(x)},
\]
from (20) and (14) we obtain the following result.
Corollary 3.1 Uniformly on compact subsets of $\mathbb{C} \setminus [-1,1], \frac{S_n(x)}{P_n^{(\alpha,\beta,\kappa_3)}(x)} = \begin{cases} \frac{\Phi(x) - \Phi(\kappa)}{\Phi(x) - 1/\Phi(\kappa)} \left[ \frac{x - \kappa}{\Phi(x) - \Phi(\kappa)} \phi'(x) - \frac{\sqrt{\kappa^2 - 1}}{\sqrt{x^2 - 1}} \right]^{-1}, & \text{if } \kappa_3 > 0, \\ \\
\frac{\Phi(x) - \Phi(\kappa)}{\Phi(x) - 1/\Phi(\kappa)} \frac{x - \kappa}{\Phi(x) - \Phi(\kappa)} \phi'(x), & \text{if } \kappa_3 > 0. 
\end{cases}

References


