A brief review of simple finite volume schemes using triangular meshes

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Abstract: This review attempts to place in perspective the variety of simple triangular discretizations which are available for constructing computational grids in order to use the Finite Volume Method. In general, there are two main schemes for simple finite volume discretizations in triangular meshes: cell-centered and vertex-centered. The two schemes differ in the location of the flux variable in the control volume with respect to the mesh. This review briefly describes some variations of the grid construction.

Keywords: Finite Volume Method, partial differential equations, conservation laws, Median Dual, Dirichlet Triangulation, Voronoi Diagrams.

1 Introduction

Numerical approaches seek appropriate forms to fulfill the Finite Volume Method (FVM) requirements. For example, numerical approaches seek to adequately establish an orthogonal segment between evaluation points and the outward normal vector, which is used in the Divergence Theorem. In general, there are two main schemes for simple finite volume discretizations in triangular meshes: cell-centered and vertex-centered. The two schemes differ in the location of the flux variable in the control volume with respect to the mesh: i) in cell-centered schemes, flux quantities are stored in the finite volume centroid themselves and the mesh has simple geometry; ii) in vertex-centered schemes, flux variables are stored in the mesh vertices; consequently, control volumes are comprised of sub-finite volumes, i.e. parts of finite volumes, which the vertex belongs.

Generally, there are two main triangular vertex-centered schemes: the Median Dual scheme for general triangular meshes and Voronoi Diagrams (with its dual, the Dirichlet Triangulation). Both create a dual mesh for determining the required quantities. In Fig. 1a, edges and faces around the central vertex are formed by dual median segments, centroid segments, and by the Dirichlet Triangulation (DT). In addition, two common techniques for simplified linear reconstruction include a Green-Gauss integration technique and the simplified least-square technique.

After this introduction, section 2 addresses the Median Dual scheme, section 3 introduces the Voronoi Diagrams and section 4 describes cell-centered control volume schemes. Subsequently, section 5 treats the Green-Gauss integration technique and section 6 deals with the simplified least-square technique. Afterwards, Section 7 discusses the vertex- and cell-centered schemes. Finally, Section 8 draws some considerations.
2 The Median Dual scheme

According to [23], one of the first formulations using triangular meshes for the generation of control volumes was proposed in [1]. The discretization by elements based on the Finite Element Method (FEM) eases the modeling of the approximated equations and normalizes the computational implementation, resulting in a better accuracy in the solution. The FEM test functions were adapted to the finite-volume discretization. Figure 1b depicts a control volume according to this scheme. The Median Dual scheme can be used with general meshes. One can show that using a specific numerical quadrature, the FEM with linear elements and the FVM on median duals are equivalent [4]. This enables convergence analysis directly from the FEM.

Although the Median Dual scheme enables generic mesh usage, consider a DT, common in the FEM, where the control volumes are generated, depicted in Fig. 1a. The geometric center of the triangle P12 in Fig. 1b is joined to the mean point of its edges $P_1$ and $P_2$. Thus, the five triangular elements, similar to the element $P_{12}$, form the control volume centered in $P$. Each triangular element contributes with two integration points, $iP_1$ and $iP_2$, in the balance conservation of the control volume centered in $P$.

There are many publications that used the Median Dual scheme, including: [2] presented aspects of unstructured grids and finite volume solvers for the Euler and Navier-Stokes Equations; [22] proposed a formulation for finite volume; [18] verified draining solutions through finite volumes; [8] simulated the injection molding; [14] proposed an adaptive edge-based unstructured finite volume formulation for the solution of biphasic flows in porous media.

Aiming towards maintaining the practicality in the mesh generation, [24] proposed to obtain control volumes similarly to [1]. Their approach is based on a mesh formed by quadrangles similarly to the generalized coordinates [23]. The advance in the field given in [24] was the coupling introduced in the equations in order to improve the convergence process. Figure 1c (from [23]) illustrates an elemental volume according to this scheme. The control volumes are obtained by joining the barycenter of the quadrangle P123 to the mean point of the edges. Quadrangular elements similar to the element P123 form the control volume centered in $P$. Furthermore, a local coordinate system is needed in order to implement the approximations. It provides more options to create the mesh and does not need to have a fixed number of points in the coordinate directions, what differs from the generalized coordinates [23]. Each element also contributes with two integration points, $iP$ and $iP_1$, in the balance conservation of the control volume centered in $P$. The arbitrary discretization of [23] consists of obtaining the control volume from a set of diamond elements limited by two centers of volumes and one area, where the equations are integrated. The element $Paib$ in Fig. 2 links the volumes centered in $P$ and 1 through the segment $P1$.

Figure 2 sketches three volumes, whose centroids are indicated by points $P$, 1 and 2. The edge between volumes $P$ and 1 is limited by points $a$ and $b$. The integration point between those volumes is indicated by point $ip$, and the vector area is indicated by $\vec{A}$. Each element $Paib$ has one integration point. The position of the integration point in each element is fundamental in order to minimize the numerical error introduced by the approximation. Points $a$ and $b$ of Fig. 2 determine the effective area where flow is changed in element $Paib$, and also is the addition of the vector areas of the segments $\vec{ap}$ and $\vec{ib}$, indicated by the vector $\vec{A}$ from the integration point $ip$. Vector $\vec{A}$ is not, necessarily, parallel to $P1$. Such area can be applied at an integration...
Figure 2: Three neighbor volumes, whose centroids are $P$, 1 and 2, whereas $ip$ is the integration point between volumes $P$ and 1.

In [23], the authors showed a formulation applied to an evolutionary convective-diffusive problem, whose velocity field $\vec{u}$ is known and the concentration $\phi$ evolves as

$$
\rho \frac{\partial \phi}{\partial t} + \vec{u} \rho \nabla \cdot \phi = \Gamma^\phi \nabla^2 \phi + S^\phi,
$$

(1)

where $\Gamma^\phi$ is the diffusion coefficient, $S^\phi$ is the source term, $\rho$ is the density and $t$ is time.

Following the FVM basic formulation for non-uniform meshes, the integration is performed over the volume named $P$. Applying the Divergence Theorem, with the source term lineraized, numerical integration in time and space yield

$$
M^n_P \phi^n_P - M^{n-1}_P \phi^{n-1}_P + \Delta t \sum_{ip=1}^3 \left[ \rho(\vec{u} \cdot \vec{A})_{ip} \phi_{ip} - \Gamma^\phi (\nabla \phi_P \cdot \vec{A})_{ip} - (S_P \phi_{ip} + S_C) \frac{\Delta V_{Pa1b}}{2} \right] = 0,
$$

(2)

where $M_P = \rho \Delta V_P$ is the mass contained in the control volume, $\vec{A}$ is the vector area of each face and $\Delta V_P$ is the area of the discrete place. When the quantity parcels of all elements are added and boundary conditions applied, there is a conservative algebraic equation of volume $P$, connected to its neighbors. Applying this scheme to all volumes results in a system of algebraic equations. When it is solved, the values $\phi$ in all centroids that comprise the discretization are obtained.

The authors of [9] adapted the scheme presented by [23] in a cell-centered approach that determinates the gradients as easy as in a vertex-centered dual scheme without generating a dual mesh. Moreover, [9] showed an approach in order to solve PDEs applying the FVM and adaptive mesh refinement pursuing to maintain an appropriate accuracy with low computational cost. The dot product between the gradient vector of the partial differential equation (PDE) dependent variable and the vector area in Eq. 2 applied in [23] was simplified in [9]. Since point 1 is on the right side of point $P$, i.e. $x_1 > x_P$ if they are at the same horizontal coordinate, vector $\vec{c} = (x_1 - x_P, y_1 - y_P)$ is generated from segment $P1$. In addition, since $\cos \alpha$ is given by

$$
\frac{\vec{c} \cdot \vec{A} = \frac{(x_1 - x_P)(y_a - y_b) + (y_1 - y_P)(x_b - x_a)}{[(x_1 - x_P)^2 + (y_1 - y_P)^2]((x_A + y_A)^2 + (x_A + y_A)^2)}^2 (\Phi_1 - \Phi_P),
$$

(3)

where $\Phi_P$ and $\Phi_1$ are values of $\phi$ stored in the respective vertices of the element $Pa1b$. This approximation depends on the direction when constructing a function in order to interpolate the derivatives of $\phi$. The direction used depends on the flux, that is previously unknown [11]. The authors in [23] wrote an interpolation function in the $\vec{A}$ direction of the element 12 and it was followed by [9]. According to [11], the calculated flux is subjected to two errors: one of
order $L^2$ because of the linear approximation of the derivative (as usual, $L^2$ is a space of 2-power integrable functions and corresponding sequence spaces [19]); and another error involving the cosines of the angles between the real flux and the vector area with the segment $PT$, which compromises the conditioning of the matrix associated with the resulting linear system.

3 Voronoi Diagrams

Voronoi Diagrams are another important vertex-centered scheme for its simplicity in the formulation. Voronoi Diagrams take advantage that the edges that comprise a control volume are orthogonal to the segment between control volume centroids (of simplices). More precisely, Voronoi Diagrams (see Fig. 1a) of a set of points $P$ are convex regions in the plane. Each region is the portion of the plane closer to one of the points of $P$ than to any other point of $P$. Figure 1d represents a Voronoi volume. It is obtained from joining the mean point of each edge to the adjacent triangle barycenters. Each segment contributes with one integration point $ip$ in the balance conservation.

The DT of a point set is defined as the dual of the Voronoi Diagrams of the set. The 2D-DT is formed by connecting two points if and only if their Voronoi regions have a common border segment. If no four or more points are cocircular, the vertices of the Voronoi Diagrams are circumcenters of the Delaunay triangles. This is true because vertices of the Voronoi represent locations that are equidistant to three (or more) sites. Because edges of the Voronoi Diagrams are the loci of points equidistant to two sites, each edge of the Voronoi Diagrams is perpendicular to the corresponding edge of the DT. This duality extends straightforwardly to 3D [2].

In summary, Voronoi Diagrams use an elegant characteristic from the DT: Voronoi volumes have edges orthogonal to the line segment between adjacent vertices and the intersection point is on the mean point of the edge that connects the two vertices. This feature is crucial in the FVM because the Divergence Theorem is based on the outward normal vector of the edge of a control volume. Examples of the wide research in Voronoi Diagrams are: [10] presented primitives for the manipulation of general subdivisions and the computation of Voronoi Diagrams; [2] presented aspects of unstructured grids and finite volume solvers for the Euler and Navier-Stokes Equations; [25] presented aspects of the Delaunay mesh generation; [12] presented an algorithm for mesh generation.

4 Cell-centered control volume schemes

In this strategy, the control volumes are the cell themselves. For instance, Fig. 3a ([3]) sketches a triangular mesh with centroids in the proper finite volumes. Figure 3b ([23]) sketches the discretization using arbitrary polygons. The finite volume approximations are performed on each edge of the polygon $P$. The integration point $ip$ in Fig. 3b is not necessarily the mean point between the volumes $P$ and 1 [23].

Figure 3: Example of meshes with control volumes in the proper finite volume.

Many publications used this strategy. Examples are: [26] applied a hexagonal mesh in the numerical simulation of mass and heat transfer in porous media; a generalization of the discretization through convex arbitrary polygons was proposed in [15] employing a versatile mesh; the authors of [5] described their approach to compute accurate solutions for time dependent fluid flows in complex geometry.
5 The Green-Gauss integration technique

A common technique for simplified linear reconstruction is the Green-Gauss linear integration technique reconstruction, where gradients are computed in specific integration points. Moreover, convective and diffusion terms are evaluated on all control volume edges.

A disadvantage of Green-Gauss reconstruction is one that occurs in many schemes, the angle evaluation between the segment between each control volume centroids and edges. Some correction should be done and numerical oscillations can occur in turbulent flows and discontinuities. Besides, special calculations, through physical and mathematical knowledge of the studied problem should be accomplished for higher-order accuracy. As an example of this strategy, [21] implemented a parallel adaptive mesh refinement (AMR) scheme for turbulent multi-phase rocket motor core flows.

6 The simplified least-square technique

The simplified least-square gradient reconstruction is completely described in [3]. The principle of the least-square reconstruction is to minimize the error in numerically approximating the integrals in the cell averages of the neighboring cells, which locally support the higher-order method [7]. Moreover, in [16] the author explains that the least-square technique may include error term weights, leading to different gradient approximations for non-linear functions. The least-square technique represent a linear function for vertex and cell-centered discretizations on arbitrary meshes. Besides the ones cited, another example of publication using this scheme is [13], which presented a conservative finite volume second-order accurate projection method on hybrid unstructured grids in steady 2D incompressible viscous recirculating flows.


7 A discussion between cell and vertex-centered control volume schemes

Determining the PDE dependent variable in the triangular volume centroids has an advantage similar to that of Median dual, that is, using finite volume centroids for generating control volumes, one can assure that the mesh will not have exterior points in the mesh. In addition, general meshes can be used. Differently from control volumes created, for instance, from Voronoi Diagrams, whose centroids are circumcenters and hence, the centroids can be exterior to the triangular volumes (because it requires DT).

In [27], the author explains that vertex-centered schemes are first-order accurate on non-smooth grids. On Cartesian or on smooth grids, the vertex-centered schemes are second or higher-order accurate depending on the flux evaluation scheme. On the other hand, the discretization error of cell-centered schemes depends strongly on the smoothness of the grid. Generally, a cell-centered scheme on triangular/tetrahedral grid leads to about two/six times more control volumes than a vertex-centered scheme. Hence, cell-centered schemes have more degrees (more unknowns) of freedom than the Median Dual scheme. In addition, control volumes in cell-centered schemes are usually smaller than those in vertex-centered schemes. This suggests that cell-centered schemes are more accurate than vertex-centered schemes. However, residual of cell-centered schemes results from much smaller number of fluxes compared to the Median Dual scheme, which is a vertex-centered scheme. Boundary condition implementation in vertex-centered schemes requires additional logic in order to assure a consistent solution at
boundary points. In the opposite, it is simple in cell-centered schemes [27]. As a result, there is no clear evidence about the better scheme. There are publications that used both cell and vertex-centered strategies. An example is [17], where the authors showed a scalable strategy for the parallelization of multiphysics unstructured mesh-iterative codes on distributed-memory systems.

8 Conclusions

In the scheme applied in [23], other directions can be used in order to construct the derivative interpolating function of the element. Specifically, a function aligned to the direction of the segment $P1$ (see Fig. 2) could be applied. A future work shall apply a hybrid scheme, where the interpolating function will depend on the flux pursuing a better approximation. Besides, the approach showed by [9] shall be adapted to use the Voronoi Diagrams in order to achieve better accuracy and low computational cost with adaptive mesh refinement.

Although the limited space, the authors hope that this review serves to introduce the ideas, principles and schemes that constitute the state-of-art in this subject. Additionally, the authors hope that the list of references and descriptions to the large body of work on this issue can provide a useful starting point for one faced with the task of constructing a nonuniform triangular mesh in order to numerically solve PDEs through the Finite Volume Method. A future survey shall cover more technical details of the schemes. In addition, it shall cover second and higher-order schemes.

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