Mathematical analysis of a third-order memristor-based Chua oscillators

Vanessa Botta, Cristiane Néspoli, Marcelo Messias
Depto de Matemática, Estatística e Computação, Faculdade de Ciências e Tecnologia, UNESP - Univ Estadual Paulista, 19060-900, Presidente Prudente, SP
E-mail: botta@fct.unesp.br, cnespoli@fct.unesp.br, marcelo@fct.unesp.br

Abstract: In this paper we present a detailed bifurcation analysis of one memristor oscillator mathematical model given by three-dimensional 5-parameter piecewise-linear system of ordinary differential equations. We show the linear analysis in the general case and we present the numerical simulations for some parameter values.

Keywords: memristor oscillator, Chua oscillator.

1 Introduction

In 2008, a team of scientists of Hewlett-Packard Company announced the fabrication of a memristor, short for memory resistor, that is a passive nonlinear two-terminal circuit element that maintains a functional relationship between the time integrals of current and voltage (see e.g. [10]). The memristor is the fourth fundamental electronic element in addition to the resistor, inductor and capacitor, that was suggested by the scientist Leon Chua in 1971 [1].

According to Stan Williams of Hewlett-Packard Labs in Palo Alto, California, “a memristor is essentially a resistor with memory”. So, for example, a computer created from memristive circuits can remember what has happened to it previously, and freeze that memory when the circuit is turned off (see e.g. [6]). Then, the memristor promises to be very useful in the fields of nanoelectronics and computer logic, for example.

In Itoh and Chua [7], we can see a basic analysis of equations of several types named nonlinear memrsitor oscillators, obtained by replacing Chuas’s diode with memristors in some of the well studied Chua’s circuits. Messias et al. [9] present a quite complete bifurcation analysis of a third-order piecewise-linear canonical oscillator, wich is one of those derived in [7] and propose another model, obtained by replacing the piecewise-linear function (wich represents the memductance) with a quadratic positive definite function.

In this paper we present a detailed bifurcation analysis of a third-order piecewise-linear memristor-based Chua oscillators, derived by Itoh and Chua [7], where the autors design a nonlinear oscillator by replacing “Chua’s diode” with an active two-terminal circuit consisting of a negative conductance and a memristor (or an active memristor).

2 A third-order memristor based Chua oscillators

As described in Itoh and Chua [7], the memristor is a passive two-terminal eletronic device described by a nonlinear constitutive relation between the device terminal voltage $v$ and the terminal current $i$, given by

\[ v = M(q)i, \quad \text{or} \quad i = W(\varphi)v. \] (1)
The functions $M(q)$ and $W(\varphi)$, which are called memresistance and memductance, are defined by

$$M(q) = \frac{d\varphi(q)}{dq} \geq 0 \quad \text{and} \quad W(\varphi) = \frac{dq(\varphi)}{d\varphi} \geq 0,$$

and represent the slope of scalar functions $\varphi = \varphi(q)$ and $q = q(\varphi)$, respectively, named the memristor constitutive relation. In spite of the relations (1) simplicity, no one has been able to propose a physical model satisfying them, until the fabrication announced in [10].

In [7], the authors give several mathematical models for memristor oscillators by replacing Chua’s diodes in some Chua’s circuits. Here we consider the Van der Pol oscillator with Chua’s diode in Fig. 1 (see [7]).

If we replace Chua’s diode with a two-terminal circuit consisting of a conductance and a flux-controlled memristor, we would obtain the circuit shown in Fig. 2.

We can represent the dynamics of this circuit by

$$\begin{align*}
C\frac{dv}{dt} &= -i - W(\varphi)v + Gv \\
L\frac{di}{dt} &= v \\
\frac{d\varphi}{dt} &= v
\end{align*},$$

where

$$W(\varphi) = \frac{dq(\varphi)}{d\varphi},$$

$$q(\varphi) = b\varphi + 0.5(a - b)(|\varphi + 1| - |\varphi - 1|).$$
The system (3) can be represented by
\[
\begin{align*}
\frac{dx}{dt} &= \alpha(-y - W(z) + \gamma x) \\
\frac{dy}{dt} &= \beta x \\
\frac{dz}{dt} &= x
\end{align*}
\] (5)
where \( x = v, \ y = i, \ z = \varphi, \ \alpha = \frac{1}{C}, \ \beta = \frac{1}{L}, \ \gamma = G \) and the piecewise-linear functions \( q(z) \) and \( W(z) \) are given by
\[
q(z) = bz + 0.5(a - b)(|z + 1| - |z - 1|),
\]
\[
W(z) = \begin{cases} 
  a, & |z| < 1 \\
  b, & |z| > 1
\end{cases},
\] (6)
respectively, where \( a, b > 0 \).

In [7], the authors present numerical simulations of system (5) solutions for the particular parameter values \( \alpha = 2, \ \beta = 1, \ \gamma = 0.3, \ a = 0.1 \) and \( b = 0.5, \) for which they found two periodic attractors. In the next section we extend the analysis made in such an article by performing a bifurcation study of system (5) when the parameters vary in the positive cone \( \{(\alpha, \beta, \gamma, a, b) \in \mathbb{R}^5 \mid \alpha, \beta, \gamma, a, b > 0\} \).

### 3 Linear Analysis

The equilibrium points of the system (5) are given by
\[
E = \{ (x, y, z) \in \mathbb{R}^3 \mid x = y = 0 \text{ and } z \in \mathbb{R} \}.
\]

The local normal stability of the equilibrium points \((0, 0, z)\) of system (5) with \(0 < a < b\) is presented in Table 1, according to the positive parameters \( \alpha, \beta, \gamma, a \) and \( b \).

<table>
<thead>
<tr>
<th>Conditions on ( \tau )</th>
<th>Conditions on ( \Delta )</th>
<th>Local stability of ((0, 0, z))</th>
<th>Cases</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\gamma}{\alpha} &lt; W(z) )</td>
<td>( \Delta &lt; 0 )</td>
<td>stable foci</td>
<td>stable foci</td>
</tr>
<tr>
<td>( \frac{\gamma}{\alpha} &gt; W(z) )</td>
<td>( \Delta &gt; 0 )</td>
<td>stable improper node</td>
<td>stable improper node</td>
</tr>
<tr>
<td>( \tau ) changes signal with ( z )</td>
<td>( \Delta &gt; 0 )</td>
<td>unstable node</td>
<td>stable node</td>
</tr>
</tbody>
</table>

Table 1: Local normal stability of the equilibria \((0, 0, z), \ |z| \neq 1\), of system (5) according to the parameter values.

**Proof.** The Jacobian matrix \( J \) of system (5) at the equilibrium point \((0, 0, z)\) is given by
\[
J = \begin{bmatrix}
\alpha(\gamma - W(z)) & -\alpha & 0 \\
-\beta & 0 & 0 \\
1 & 0 & 0
\end{bmatrix},
\] (7)
from which follows that this equilibrium has the eigenvalues \( \lambda_1 = 0 \) and \( \lambda_{2,3} \) given by the solutions of the quadratic equation

\[
\lambda^2 - \alpha (\gamma - W(z)) \lambda + \alpha \beta = 0.
\] (8)

Assume that \( \tau = \alpha (\gamma - W(z)) \), \( D = \alpha \beta \) and \( \Delta = \tau^2 - 4D \), then the eigenvalues are given by

\[
\lambda_1 = 0 \quad \text{and} \quad \lambda_{2,3} = \frac{\tau \pm \sqrt{\tau^2 - 4D}}{2},
\]

with the corresponding eigenvectors

\[
v_1 = (0, 0, 1), \quad v_2 = \left( \frac{\tau - \sqrt{\Delta}}{2}, \frac{\tau^2 - \Delta}{4\alpha}, 1 \right)
\]

and

\[
v_3 = \left( \frac{\tau + \sqrt{\Delta}}{2}, \frac{\tau^2 - \Delta}{4\alpha}, 1 \right).
\]

It remains to verify the normal hyperbolicity of the equilibria \((0, 0, z)\). In order to do that it is enough to calculate the equation of the plane \( \pi \) spanned by the eigenvectors \( v_1 \) and \( v_2 \) above. If \( \lambda_1 \) and \( \lambda_2 \) are real eigenvalues, then such plane has \( n = v_2 \wedge v_3 \) as its normal vector. Calculating the scalar product of \( n \) with the vector \((0, 0, 1)\) we obtain

\[
\langle n, (0, 0, 1) \rangle = -\beta \sqrt{\Delta},
\] (9)

which implies that the plane \( \pi \) is transversal do the \( z \)-axis, except in the cases in which the parameters satisfy the condition \( \Delta = 0 \).

Then by the Stable Manifold Theorem the invariant manifolds (stable and unstable) associated to the equilibria \((0, 0, z)\) with \(|z| \neq 1\) are tangent to the plane \( \pi \), hence they are transversal to the \( z \)-axis. Therefore, the equilibria are normally hyperbolic, except when \( \Delta = 0 \). In the case of complex eigenvalues we get the same type of result by considering the plane spanned by \( \text{Re}(v_1) \) and \( \text{Im}(v_1) \).

Finally, analyzing the possibilities for the eigenvalues (3) according to the relations between \( \tau \) and \( D \), which depend on the parameters \( \alpha, \beta, \gamma, a, b \), we have the cases described in Table 1 for the (normal) local stability of the equilibria \((0, 0, z)\), with \(|z| < 1\) and \(|z| > 1\). This ends the proof of Theorem 3.

The case \( 0 < b < a \) can be treated in the same way.

From Theorem 3 we can obtain regions in the \( 5 \)-dimensional parameter space for which all the equilibria \((0, 0, z)\) of system (5) are locally normally stable (cases (a), (b) and (c) of Table 1) and regions in which they are locally unstable (cases (d), (e) and (f)). On the other hand, there are regions in which, for a fixed set of the parameter values inside them, changes in the local stability of the equilibria occur (cases (g), (h) and (i)). It is a type of bifurcation from a line of equilibria without varying the parameter values, i.e. a “bifurcation without parameter”, which has yet been described in the literature and mentioned before (see e.g. [4]).

The change in the local stability of an isolated equilibrium point of and ODE system is known to lead to important topological changes in its phase space, which are called bifurcations of the solutions. In the particular case in which the real part of a pair of complex conjugate eigenvalues cross transversely the imaginary axes as a parameter is varied, under certain conditions a limit cycle may be created, leading to oscillations of the solutions. It is the well known Hopf bifurcation.
4 Numerical Simulations

In this section we present the solutions of system (5) for some parameter values using the software Mathematica 7.0.

Fig. 3 shows the local (normal) stability of the equilibria \((0,0,z)\) where \(\tau < 0\) and \(\Delta < 0\) (case (a) of Table 1).

Fig. 4 shows the local (normal) instability of the equilibria \((0,0,z)\) where \(\tau > 0\) and \(\Delta < 0\) (case (d) of Table 1).

Now we consider system (5) with the same parameter values used in [7], with \(\alpha = 2, \beta = 1, \gamma = 0.3, a = 0.1\) and \(b = 0.5\). We can see a periodic attractor in Fig. 5 (case (d) of Table 1).

The next figures show the local (normal) instability (Fig. 6) and stability (Fig. 7) of the
equilibria \((0, 0, z)\) where \(\tau\) changes the signal with \(z\) and \(\Delta < 0\) (case (g) of Table 1).

![Figure 6: Solutions of system (5) with \(\alpha = 1, \beta = 1, \gamma = 0.2, a = 0.02, b = 2\) and \(|z| < 1\).](image)

![Figure 7: Solutions of system (5) with \(\alpha = 1, \beta = 1, \gamma = 0.2, a = 0.02, b = 0.5\) and \(|z| > 1\).](image)

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**References**


