Robustness on Intuitionistic Fuzzy Connectives

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Abstract: The main contribution of this paper is concerned with the robustness of intuitionistic fuzzy connectives in fuzzy reasoning. Starting with an evaluation of the sensitivity in n-order functions on the class of intuitionistic fuzzy sets, we apply the results in the intuitionistic (S,N)-implication class. The paper formally states that the robustness preserves the projection functions in such class.

Keywords: Robustness Analysis, Fuzzy Reasoning, Intuitionistic Fuzzy Logic

1 Introduction

Robustness or sensitivity can be conceived as a fundamental property of a logical system stating that the conclusions are not essentially changed if the assumed conditions varied within reasonable parameters. Significant works have been developed in this research area, see, e.g. [6, 11, 13, 16, 15]. Moreover, because fuzzy connectives (mainly t-norms and t-conorms, implications and coimplications) are important elements in the fuzzy reasoning, the corresponding investigation of the δ sensitivity in such fuzzy connectives, in terms of [12], will be carried out in this work, following previous work, see [14]. In addition, we extend the notion of sensitivity towards intuitionistic fuzzy connectives, based on the study proposed in [1]. We aim to contribute to the sensitivity interpretation related to truth and non-truth in conditional fuzzy rules based on Intuitionistic Fuzzy Logic (IFL).

The next section describes the basic concepts of fuzzy connectives and intuitionistic fuzzy connectives. The sensitivity of fuzzy connectives and general results of robustness of intuitionistic fuzzy connectives are stated in Sections 3 and 4, respectively. Final remarks are reported in the conclusion.

2 Preliminaries

2.1 Fuzzy connectives

Notions concerning t-(co)norms, (co)implications and dual functions are reported based on [9] and [10].

2.1.1 Fuzzy negations

Let $U = [0, 1]$ be the unit interval of real numbers. Recall that a function $N : U \to U$ is a fuzzy negation if it verifies the properties: $N_1 : N(0)=1$ and $N(1)=0$; $N_2 :$ If $x \geq y$ then $N(x) \leq N(y)$, $\forall x, y \in U$. Fuzzy negations satisfying the involutive property:

$N_3$ $N(N(x)) = x, \forall x \in U$,

are called strong negations, e.g. the standard negation $N_S(x) = 1 - x$. When $x = (x_1, x_2, \ldots, x_n) \in U^n$ and $N$ is a fuzzy negation, we use the denotation:

$$N(x) = N^n(x_1, x_2, \ldots, x_n) = (N(x_1), N(x_2), \ldots, N(x_n))$$
Let \( N \) be a negation. The \( N \)-dual function of \( f : U^n \rightarrow U \) is given by:

\[
f_N(x_1, \ldots, x_n) = N(f(N(x_1), \ldots, N(x_n))).
\]

Notice that, when \( N \) is involutive, \( (f_N)_N = f \), that is the \( N \)-dual of \( f_N \) is the function \( f \). In addition, when \( f = f_N \), it is clear that \( f \) is a self-dual function.

### 2.1.2 Triangular (co)norms

A binary function \((S)T : U^2 \rightarrow U\) is a \( t\)-(co)norm (triangular (co)norm) if and only if it satisfies the boundary conditions together with the commutative, associative and monotonic properties, and has an neutral element \((S(x,0) = x)\ T(x,1) = x\), for all \( x \in U \).

Let \( N \) be a fuzzy negation on \( U \). And, the mappings \( T_N, S_N : U^2 \rightarrow U \) denoting the \( N \)-dual functions of a \( t \)-norm \( T \) and of a \( t \)-conorm \( S \), respectively defined as:

\[
T_N(x,y) = N(T(N(x), N(y))), \quad S_N(x,y) = N(S(N(x), N(y))).
\]

### 2.1.3 Fuzzy (co)implications

A (co)implication operator \((J)I : U^2 \rightarrow U\) extends the classical (co)implication function, i.e., it satisfies the boundary conditions:

\[
\begin{align*}
I_0 & : I(1,1) = I(0,1) = I(0,0) = 1, I(1,0) = 0; \quad J_0 : I(1,1) = I(0,1) = I(0,0) = 1, I(1,0) = 0. \\
I_1 & : I(x,y) \geq I(z,y) \text{ if } x \leq z; \quad J_1 : I(x,y) \geq I(z,y) \text{ if } x \leq z; \\
I_2 & : I(x,y) \leq I(x,z) \text{ if } y \leq z; \quad J_2 : I(x,y) \leq I(x,z) \text{ if } y \leq z; \\
I_3 & : I(0,x) = 1; \quad J_3 : I(0,x) = 1; \\
I_4 & : I(x,1) = 1; \quad J_4 : J(0,y) = 0; \\
I_5 & : I(1,0) = 0; \quad J_5 : J(1,0) = 0.
\end{align*}
\]

There exist many classes of (co)implication functions (see, e.g., [9] and [5]). We consider in this paper the \((S,N)\)-implications defined by

\[
I_{S,N}(x,y) = S(N(x), y), \quad \forall x, y \in U,
\]

when \( S \) is a \( t \)-conorm and \( N \) is a fuzzy negation. If \( N \) is a strong fuzzy negation, then \( I_{S,N} \) is called a strong implication or \((S,N)\)-implication. When \( S_N \) is the \( N \)-dual function of the \( t \)-conorm \( S \), the corresponding \( N \)-dual functions are \( S\)-coimplications given by

\[
(I_{S,N})_N(x,y) = S_N(N(x), y), \quad \forall x, y \in U,
\]

### 2.2 Intuitionistic Fuzzy Connectives

According with [2], an intuitionistic fuzzy set (IFS) \( A \) in a non-empty, finite universe \( \chi \), expressed as

\[
A = \{(x, \mu_A(x), \nu_A(x))| x \in \chi, \mu_A(x) + \nu_A(x) \leq 1\},
\]

generalizes a fuzzy set \( A = \{(x, \mu_A(x), 1 - \mu_A(x))| x \in \chi\} \), since \( \nu_A(x) \), which is the non-membership degree of an element \( x \), is less than or equal to the complement of the related interval membership degree \( \mu_A(x) \), and therefore is not necessarily equal to this one.

Let \( \tilde{U} = \{\tilde{x} = (x_1, x_2)|(x_1, x_2) \in U^2, x_1 \leq N_S(x_2)\} \) be the set of all intuitionistic fuzzy sets and \( l_{\tilde{U}}, r_{\tilde{U}} : \tilde{U} \rightarrow U \) be the projection functions on \( \tilde{U} \), which are given by \( l_{\tilde{U}}(\tilde{x}) = l_{\tilde{U}}(x_1, x_2) = x_1 \) and \( r_{\tilde{U}}(\tilde{x}) = r_{\tilde{U}}(x_1, x_2) = x_2 \), respectively.

Thus, for all \( \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n) \in \tilde{U}^n \), such that \( \tilde{x}_i = (x_{i1}, x_{i2}) \) and \( x_{i1} = N_S(x_{i2}) \) when \( 1 \leq i \leq n \), let \( l_{\tilde{U}^n}, r_{\tilde{U}^n} : \tilde{U}^n \times \tilde{U}^n \rightarrow \tilde{U}^n \) be projections given by:

\[
l_{\tilde{U}^n}(\tilde{x}) = (l_{\tilde{U}}(\tilde{x}_1), l_{\tilde{U}}(\tilde{x}_2), \ldots, l_{\tilde{U}}(\tilde{x}_n)) = (x_{11}, x_{21}, \ldots, x_{1n})
\]

\[
r_{\tilde{U}^n}(\tilde{x}) = (r_{\tilde{U}}(\tilde{x}_1), r_{\tilde{U}}(\tilde{x}_2), \ldots, r_{\tilde{U}}(\tilde{x}_n)) = (x_{12}, x_{22}, \ldots, x_{2n}).
\]

Consider also the order relation \( \tilde{x} \leq \tilde{y} \Leftrightarrow x_1 \leq y_1 \) and \( x_2 \geq y_2 \) such that \( 0 = (0,1) \leq \tilde{x} \) and \( \tilde{1} = (1,0) \geq \tilde{x} \), for all \( \tilde{x}, \tilde{y} \in \tilde{U} \). [1]
2.2.1 Intuitionistic fuzzy negations

An intuitionistic fuzzy negation (IFN shortly) \( N_I : \tilde{U} \rightarrow \tilde{U} \) verifies, for all \( \tilde{x}, \tilde{y} \in \tilde{U} \) the properties:

\( N_I 1: N_I(0) = N_I(0, 1) = \tilde{1} \) and \( N_I(1) = N_I(1, 0) = 0; \)

\( N_I 2: \) If \( \tilde{x} \geq \tilde{y} \) then \( N_I(\tilde{x}) \leq N_I(\tilde{y}). \)

In addition, \( N_I \) is a strong IFN if it is also involutive:

\( N_I 3: N_I(N_I(\tilde{x})) = \tilde{x}, \forall \tilde{x} \in \tilde{U}. \)

Consider \( N_I \) as IFN in \( \tilde{U} \) and \( \tilde{f} : \tilde{U}^n \rightarrow \tilde{U}. \) For all \( \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n) \in \tilde{U}^n \), the \( N_I \)-dual intuitionistic function of \( \tilde{f} \), denoted by \( \tilde{f}_{N_I} : \tilde{U}^n \rightarrow \tilde{U} \), is given by:

\[
\tilde{f}_{N_I}(\tilde{x}) = N_I(\tilde{f}(N_I(\tilde{x}_1), \ldots, N_I(\tilde{x}_n))).
\]  

(7)

Let \( \tilde{N}_I \) be a SIFN, \( \tilde{f} \) is a self-dual intuitionistic function.

By [3, Theorem 1] [8, 7], a function \( N_I : \tilde{U} \rightarrow \tilde{U} \) is a strong intuitionistic negation iff there exists a strong negation \( N : U \rightarrow U \) such that it is expressed as:

\[
N_I(\tilde{x}) = (N(N_S(x_2), N_S(N(x_1))).
\]  

(8)

2.2.2 Intuitionistic t-(co)norms

A function \((S_I,T_I) : \tilde{U}^2 \rightarrow \tilde{U}\) is a fuzzy triangular (co)norm (t-(co)norm shortly), if it is a commutative, associative and increasing function with neutral element \((\tilde{0}, \tilde{1})\).

Based on results of [3, Definition 3], an intuitionistic t-norm \((S_I,T_I) : \tilde{U}^2 \rightarrow \tilde{U}\) is t-representable if there exists a t-norm \(T' : U^2 \rightarrow U\) and a t-conorm \(S' : U^2 \rightarrow U\) such that \(T'(x,y) \leq N_S(S(N_S(x), N_S(y)))\) and, for all \(\tilde{x} = (x_1, x_2), \tilde{y} = (y_1, y_2) \in U\), it is expressed as:

\[
S_I(\tilde{x}, \tilde{y}) = S_I((x_1, x_2), (y_1, y_2)) = (S(x_1, y_1), T(x_2, y_2));
\]  

(9)

\[
T_I(\tilde{x}, \tilde{y}) = T_I((x_1, x_2), (y_1, y_2)) = (T(x_1, y_1), S(x_2, y_2)).
\]  

(10)

2.2.3 Intuitionistic fuzzy implications

An intuitionistic fuzzy implication \( I_I : \tilde{U}^2 \rightarrow \tilde{U} \) is a function satisfying the boundary conditions:

\( I_I 0 \; I_I(0, 0) = I_I(0, \tilde{1}) = I_I(\tilde{1}, 0) = \tilde{1} \) and \( I_I(\tilde{1}, \tilde{0}) = 0; \)

Definition 2 [5, Definition 3] An intuitionistic fuzzy implication \( I_I : \tilde{U}^2 \rightarrow \tilde{U} \) is a function such that \( I_I(\tilde{1}, \tilde{0}) = \tilde{0} \) and, for all \( \tilde{x}, \tilde{y}, \tilde{z} \in \tilde{U} \), the following properties hold:

\( I_I 1: \tilde{x} \leq \tilde{z} \Rightarrow I_I(\tilde{x}, \tilde{y}) \geq I_I(\tilde{z}, \tilde{y}); \) (1-place antimonotonicity);

\( I_I 2: \tilde{y} \leq \tilde{z} \Rightarrow I_I(\tilde{x}, \tilde{y}) \leq I_I(\tilde{x}, \tilde{z}); \) (2-place monotonicity);

\( I_I 3: I_I(\tilde{0}, \tilde{y}) = \tilde{1} \) (dominance falsity);

\( I_I 4: I_I(\tilde{x}, \tilde{1}) = \tilde{1} \) (boundary condition);

\( I_I 5: \) If \( x_1 + x_2 = 1 \) and \( y_1 + y_2 = 1 \) then \( \pi I_I((x_1, x_2), (y_1, y_2)) = 0. \)

Recovering Definition 1 of a fuzzy implication in the sense of J. Fodor and M. Roubens’ work [10], Definition 2 also reproduces fuzzy (co)implications if, for all \( \tilde{x} = (x_1, x_2), \tilde{y} = (y_1, y_2) \in \tilde{U} \) we have \( x_1 = N_S(x_2) \) and \( y_1 = N_S(y_2) \). According to [1] and [8], another way of defining the Atanassov’s operator \( I_I \) is to consider boundary conditions in \( I_I 0 \) and properties \( I_I 1 \) and \( I_I 2 \).

Based on [3, Theorem 4], a function \( I_I : \tilde{U}^2 \rightarrow \tilde{U} \) is a representable intuitionistic \((S, N)\)-implication based on a strong negation \( N_I : \tilde{U} \rightarrow \tilde{U} \) iff there exist \((S, N)\)-implications \( I_a, I_b : U^2 \rightarrow U \), such that for all \( \tilde{x} = (x_1, x_2), \tilde{y} = (y_1, y_2) \in U, I_I \) is expressed as:

\[
I_I(\tilde{x}, \tilde{y}) = (I_a(N_S(x_2), y_1), N_S(I_b(x_1, N_S(y_2))).
\]  

(11)
3 The pointwise sensitivity of fuzzy connectives

In the following, based on [12] and [14], the study of a δ sensitivity of an n-order function \( f \) at point \( x \) (or a pointwise sensitivity) on the domain \( U \) is considered, mainly related to the class of \((S, N)\)-implications.

**Definition 3** [12, Definition 1] Let \( f : U^n \to U \) be an n-order function, \( \delta \in U \) and \( x = (x_1, x_2, \ldots, x_n) \), \( y = (y_1, y_2, \ldots, y_n) \in U^n \). The δ sensitivity of \( f \) at point \( x \), denoted by \( \Delta_f(x, \delta) \), is defined by

\[
\Delta_f(x, \delta) = \sup \{ |f(x) - f(y)| : |x_i - y_i| \leq \delta, 1 \leq i \leq n \}
\]

Now, we investigate the δ sensitivity in fuzzy connectives, in terms of Definition 3. The δ-sensitivity at point \( x \in U^n \) of binary functions based on the monotonicity property in their both arguments are reported.

**Proposition 1** [12, Theorem 2] Let \( f : U \to U \) be a reverse order function, such that \( x \leq y \Rightarrow f(x) \geq f(y) \), \( \delta \in U \) and \( x \in U \). The sensitivity of \( f \) at point \( x \) is given by

\[
\Delta_f(x, \delta) = |f(x) - f(x + \delta) \land 1| \lor [f((x - \delta) \lor 0) - f(x)].
\]

In particular, Eq. (13) holds for a fuzzy negation function.

Henceforth, in order to make denotational easier, when \( f : U^2 \to U \) and \( x = (x, y) \in U^2 \), consider the following notations:

\[
\begin{align*}
\hat{f}[x] &\equiv f((x - \delta) \lor 0, (y - \delta) \lor 0); \\
\hat{f}[x] &\equiv f((x - \delta) \lor 0, (y + \delta) \land 1); \\
\check{f}[x] &\equiv f((x + \delta) \land 1, (y - \delta) \lor 0); \\
\check{f}[x] &\equiv f((x + \delta) \land 1, (y + \delta) \land 1).
\end{align*}
\]

**Proposition 2** [12, Theorem 1] Consider \( f : U^2 \to U \), \( \delta \in U \) and \( x = (x, y) \in U^2 \).

(i) If \( f \) is a monotone function, which means, \( x \leq x', y \leq y' \Rightarrow f(x, y) \leq f(x', y') \) then it follows that

\[
\Delta_f(x, \delta) = (f(x) - f[x]) \lor (f[x] - f(x))
\]

(ii) If \( f \) verifies both properties, 1-place antimonotonicity and 2-place monotonicity, then we have:

\[
\Delta_f(x, \delta) = (f(x) - f[x]) \lor (f[x] - f(x))
\]

Based on [12], the next proposition is presented as a consequence of Proposition 2:

**Proposition 3** [12, Corollary 1] Let \( T, S \) and \( I_{S,N} \) be a t-norm, t-conorm and an \((S, N)\)-implication. When \( x \in U^2 \) and \( \delta \in U \), the statements in the following hold:

(i) the sensitivity of a t-norm \( T \) and a t-conorm \( S \) at point \( x \), respectively, are both defined by Eq. (14);

(ii) the sensitivity of an \((S, N)\)-implication \( I_{S,N} \) at point \( x \), is defined by Eq. (15).

3.1 The pointwise sensitivity of \( N \)-dual fuzzy connectives

In the preceding section we described the definitions as foundations to study the pointwise sensitivity of \( N \)-dual fuzzy connectives.

**Proposition 4** [14, Proposition 6] Let \( f : U^2 \to U \) be a second-order function, \( x = (x, y) \in U^n \) and \( N \) be the standard fuzzy negation. The following equations hold:

(i) \( f_N[x] = N(f([N(x)]) \)

(ii) \( f_N[x] = N(f([N(x)]) \)

(iii) \( f_N[x] = N(f([N(x)]) \)

(iv) \( f_N[x] = N(f([N(x)]) \)

Taking a strong fuzzy negation \( N \), Proposition 5 states that the sensitivity of a n-order function \( f \) at a point \( x \) is equal to the sensitivity of its dual function \( f_N \).
Proposition 5  [14, Theorem 1] Consider \( f : U^2 \to U \), \( \delta \in \hat{U} \) and \( x = (x, y) \in U^2 \). Let \( \Delta_f(x, \delta) \) be the sensitivity of \( f \) at point \( x \). If \( N \) is the standard fuzzy negation (\( N = N_S \) in Eq (1)) and \( f_N \) is the \( N \)-dual function of \( f \) then the sensitivity of \( f_N \) at point \( x \) is given by

\[
\Delta_{f_N}(x, \delta) = \Delta_f((N(x)), \delta) \tag{16}
\]

Proposition 6  [14, Prop. 7] Let \( f_N \) be the standard fuzzy negation. \( f_N \) is the \( N \)-dual function related to \( f : U^2 \to \hat{U} \), \( \delta \in U \) and \( x = (x, y) \in U^2 \). The sensitivity of \( f_N \) at point \( x \) is given by the following cases:

(i) if \( f \) is increasing with respect to its variables then:

\[
\Delta_{f_N}(x, \delta) = (f_N[x] - f_N(x)) \lor (f_N[x] - f_N(x)) \tag{17}
\]

(ii) if \( f \) is decreasing with respect to its variable and increasing with its second variable then

\[
\Delta_{f_N}(x, \delta) = (f_N[x] - f_N(x)) \lor (f_N[x] - f_N(x)) \tag{18}
\]

Proposition 7  [14, Prop.8] Let \( (T)_N, (S)_N, (I_{S,N})_N \) be \( N \)-dual functions related to a t-norm \( T \), a t-conorm \( S \) and an implication \( I_{S,N} \), respectively. If \( x \in U^2 \) and \( \delta \in U \), the statements as follows hold:

(i) \( \Delta_{(T)_N}(x, \delta) \) and \( \Delta_{(S)_N}(x, \delta) \) are both defined by Eq. (17);

(ii) \( \Delta_{I_{S,N}}(x, \delta) \) is defined by Eq. (18).

Proposition 8  [14, Prop.10] Let \( N \) be the standard fuzzy negation and \( f_N : U^2 \to \hat{U} \) be the \( N \)-dual function of a function \( f : U^2 \to U \). Then the maximum sensitivities of \( f \) and \( f_N \) are related by Eq. (19):

\[
\Delta_{f_N}(\delta) = \Delta_f(\delta) \tag{19}
\]

4 The pointwise sensitivity of intuitionistic fuzzy connectives

In the following, the study of a \( \delta \) sensitivity of an intuitionistic t-(co)norm at point \( x \) on the domain \( \hat{U}^2 \) is considered in order to extend the work introduced in [12] to the class of intuitionistic fuzzy connectives which are representable by fuzzy connectives.

Definition 4 Let \( f : \hat{U}^n \to \hat{U} \) be an \( n \)-order function, \( \delta = (\delta_1, \delta_2) \in U^2 \) and \( \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n), \tilde{y} = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n) \in \hat{U}^n \). The \( \delta \) sensitivity of \( f \) at point \( \tilde{x} \), denoted by \( \Delta_f(\tilde{x}, \delta) \), is defined by

\[
\Delta_f(\tilde{x}, \delta) = \sup\{|f(\tilde{x}) - f(\tilde{y})| : |\tilde{x}_j - \tilde{y}_j|_{i \in \{1,2,\ldots,n\}, j \in \{1,2\}} \leq \delta_j\} \tag{20}
\]

The following propositions state that pointwise sensitivity is preserved by the projection functions applied to intuitionistic connectives that are t-representable in the same sense of [3, 8, 7].

Proposition 9 Let \( N_I : \hat{U} \to \hat{U} \) be a representative intuitionistic negation as defined by Eq. (8). When \( \delta = (\delta_1, \delta_2) \in U^2 \) and \( \tilde{x} \in \hat{U}^2 \), the \( \delta \) sensitivity of \( (S_I)_T \) at point \( \tilde{x} \), is defined by

\[
(\Delta_{N_I}(\tilde{x}, \delta) = (\Delta_N(l_{\hat{U}^2}(\tilde{x}), \delta), \Delta_N(r_{\hat{U}^2}(\tilde{x}), \delta))) \tag{21}
\]

Proof: Straightforward from Definition 4 and Proposition 1.

Proposition 10 Let \( (S_I)_T \) : \( \hat{U}^2 \to \hat{U} \) be a representative intuitionistic t-(co)norm as defined by (Eq. (9)) Eq. (10). Consider \( \delta = (\delta_1, \delta_2) \in U^2 \) and \( \tilde{x} \in \hat{U}^2 \). The \( \delta \) sensitivity of \( (S_I)_T \) at point \( \tilde{x} \), denoted by \( (\Delta_{S_I}(\tilde{x}, \delta)) \Delta_{T_I}(\tilde{x}, \delta) \), is defined by

\[
(\Delta_{S_I}(\tilde{x}, \delta) = (\Delta_S(l_{\hat{U}^2}(\tilde{x}), \delta), \Delta_T(r_{\hat{U}^2}(\tilde{x}), \delta)), \Delta_{T_I}(\tilde{x}, \delta) = (\Delta_T(l_{\hat{U}^2}(\tilde{x}), \delta), \Delta_S(r_{\hat{U}^2}(\tilde{x}), \delta)) \tag{22}
\]
The main contribution of this paper is concerned with the study of robustness on intuitionistic operators of fuzzy inference dependent on considering interval-valued fuzzy rules based on interval-valued
reduce sensitivity in the corresponding fuzzy connectives.

Proof: For all \( \bar{x}, \bar{y} \in U^2 \) given as \( \bar{x} = (\bar{x}_1, \bar{x}_2) \), such that \( \bar{x}_1 = (x_{11}, x_{12}) \), \( x_{11} \leq N_S(x_{12}) \) and \( \bar{x}_2 = (x_{21}, x_{22}) \), \( x_{21} \leq N_S(x_{22}) \); \( \bar{y} = (\bar{y}_1, \bar{y}_2) \), such that \( \bar{y}_1 = (y_{11}, y_{12}) \), \( y_{11} \leq N_S(y_{12}) \) and \( \bar{y}_2 = (y_{21}, y_{22}) \), \( y_{21} \leq N_S(y_{22}) \). It holds that:

\[
\Delta_{T_I}(\bar{x}, \delta) = \sup\{|T_I(\bar{x}) - T_I(\bar{y})| : |\bar{x}_{ji} - \bar{y}_{ji}|_{i,j \in \{1,2\}} \leq \delta\} \text{ by Eq. (20)}
\]

Therefore, it follows that \( l_{U^2}(\Delta_{T_I}(\bar{x}, \delta)) = \Delta_{T}(l_{U^2}(\bar{x}), \delta) \) and \( r_{U^2}(\Delta_{T_I}(\bar{x}, \delta)) = \Delta_{S}(r_{U^2}(\bar{x}), \delta) \). In analogous way, it can be proved that for \( \delta \) sensitivity of \( S_I \) at point \( \bar{x} \) it holds that \( l_{U^2}(\Delta_{S_I}(\bar{x}, \delta)) = \Delta_{S}(l_{U^2}(\bar{x}, \delta)) \) and \( r_{U^2}(\Delta_{S_I}(\bar{x}, \delta)) = \Delta_{S}(r_{U^2}(\bar{x}, \delta)) \).

We now study the \( \delta \) sensitivity of an intuitionistic \((S, N)\)-implication at point \( x \) on the domain \( U^2 \).

**Proposition 11** Let \( I_I : U^2 \rightarrow U \) be a t-representable intuitionistic \((S, N)\)-implication as defined by Eq.(11). Consider \( \delta = (\delta_1, \delta_2) \in U^2 \) and \( \bar{x} \in U^2 \). The \( \delta \) sensitivity of \( I_I \) at point \( \bar{x} \) is defined by

\[
\Delta_{I_I}(\bar{x}, \delta) = (\Delta_{I_a}(l_{U^2}(\bar{x}), \delta), \Delta_{I_b}(r_{U^2}(\bar{x}), \delta))
\]

Proof: Let \( I_I \) be a representable \((S, N)\)-implication obtained by fuzzy \((S, N)\)-implications \( I_a, I_b \) and the standard fuzzy negation \( N_S \), as defined by Eq.(11). For all \( \bar{x}, \bar{y} \in U^2 \), it holds that:

\[
\Delta_{I_I}(\bar{x}, \delta) = \sup\{|I_I(\bar{x}) - I_I(\bar{y})| : |\bar{x}_{ji} - \bar{y}_{ji}|_{i,j \in \{1,2\}} \leq \delta\} \text{ by Eq. (20)}
\]

Therefore, it follows that \( l_{U^2}(\Delta_{I_I}(\bar{x}, \delta)) = \Delta_{I_a}(l_{U^2}(\bar{x}), \delta_1) \) and \( r_{U^2}(\Delta_{I_I}(\bar{x}, \delta)) = \Delta_{I_b}(r_{U^2}(\bar{x}), \delta_2) \).

**5 Conclusion**

The main contribution of this paper is concerned with the study of robustness on intuitionistic operators mainly used in fuzzy reasoning based on IFL. Taking the class of strong fuzzy negation (standard negation), the paper formally states that the sensitivity of an \( n \)-order intuitionistic fuzzy connective at a point \( x \in U^n \) preserves its projections related to the sensitivity of its fuzzy approach at the same point.

Therefore, when these sensitivities are used in the inference process of a fuzzy rule system based on intuitionistic fuzzy connectives, the work of estimating their sensitivity to small changes is related to reduce sensitivity in the corresponding fuzzy connectives.

Based on previous work on the study of fundamental properties of interval-valued fuzzy (co)implications (see, e.g. [4]), our current investigation aims clearly to contemplate two approaches: (i) the sensitivity of fuzzy inference dependent on considering interval-valued fuzzy rules based on interval-valued
fuzzy connectives; (ii) the extension of the robustness studies in order to consider other main classes of (co)implications: R-(co)implications and QL-(co)implications.

References


