Extension of Negations Generated from T-norms on Bounded Lattices

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Resumo: Given a t-norm T on a complete lattice L it is possible to define a negation NT which is called the negation generated by T. In this paper it presented a way to extend this class of negations defined on a sublattice and considering a relaxed notion of sublattice.

Palavras-chave: Extension, T-norm, Negation, Complete Lattice

1 Introduction

In general, the task of extending functions is not simple. It is necessary, in most cases, to impose some properties to the function and their domain. As, in particular, t-norms, t-conorms and fuzzy negations are functions, it is pertinent to question under what conditions a t-norm (t-conorm and fuzzy negation) can be extended from a sublattice M to a lattice L.

One of the first works that have presented a way to extend t-norms was published in 2008 by Saminger-Platz et. al [13]. In this direction but considering a more general framework for the concept of sublattice, Palmeira and Bedregal in [11] have presented a different way to extend t-norms, t-conorms and fuzzy negations that generalizes the extension proposed in [13]. In [11], given arbitrary bounded lattices M and L it said that M is a sublattice of L if it is a retract of L, i.e. if there are a retraction r : L → M and a pseudo-inverse s : M → L (see definition 1.3) such that r ◦ s = idM (see figure 1).

In this paper it is applied the method of extending fuzzy operators proposed in [11, 12] to extend a particular class of fuzzy negations, namely the class of negations NT generated by a given t-norm T. In addition, we prove some results related to these extensions.

1.1 Bounded and Complete Lattices

In this subsection we define some useful concepts on bounded lattices which are based in the papers [1, 2, 11]. If the reader needs a deeper text on lattice theory we suggest the books [3, 4, 5, 7, 8, 9].

Let L be a nonempty set. If ∧L and ∨L are two binary operations on L, then (L, ∧L, ∨L) is a lattice provided that for each x, y, z ∈ L, the following properties hold:

1. x ∧L y = y ∧L x and x ∨L y = y ∨L x;

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2. \((x \land y) \land z = x \land (y \land z)\) and \((x \lor y) \lor z = x \lor (y \land z)\);

3. \(x \land (x \lor y) = x\) and \(x \lor (x \land y) = x\).

If in \((L, \land, \lor)\) there are elements 0 and 1 such that, for all \(x \in L\), \(x \land 1 = x\) and \(x \lor 0 = x\), then \((L, \land, \lor, 0, 1)\) is called a bounded lattice.

Moreover, it is very known that, given a lattice \(L\), the relation defines a partial order on \(L\). This order will be used by us to compare elements.

Recall also that a lattice \(L\) is called a complete lattice if every subset of it has a top and a bottom element.

Throughout this paper we take \(L\) as a bounded lattice as defined above. If \(L\) represents another thing, the appropriate distinction will be made.

**Definition 1.1** Let \((L, \land, \lor, 0_L, 1_L)\) and \((M, \land, \lor, 0_M, 1_M)\) be bounded lattices. A mapping \(f : L \to M\) is said to be a lattice homomorphism if, for all \(x, y \in L\), we have

1. \(f(x \land y) = f(x) \land_M f(y)\);
2. \(f(x \lor y) = f(x) \lor_M f(y)\);
3. \(f(0_L) = 0_M\) and \(f(1_L) = 1_M\).

**Remark 1.1** Recall that, an injective (a surjective) lattice homomorphism is called a monomorphism (epimorphism) and a bijective lattice homomorphism is called an isomorphism. An automorphism is an isomorphism from a lattice to itself.

**Proposition 1.1** Every lattice homomorphism preserves the order.

**Proof:** Let \(f : L \to M\) be a lattice homomorphism. Since \(x \leq_L y\) if only if \(x \land_L y = x\), therefore \(f(x) = f(x \land_L y) = f(x) \land_M f(y)\) and hence \(f(x) \leq_M f(y)\).

**Proposition 1.2** Let \(L\) be a bounded lattice. Then a function \(f : L \to L\) is an automorphism if and only if

1. \(f\) is bijective and
2. \(x \leq_L y\) if and only if \(f(x) \leq_L f(y)\).

**Proof:** See [11].

### 1.2 Retracts and sublattices

**Definition 1.2** A homomorphism \(r\) of a lattice \(L\) onto a lattice \(M\) is said to be a retraction if there exists a homomorphism \(s\) of \(M\) into \(L\) which satisfies \(r \circ s = \text{id}_M\). A lattice \(M\) is called a retract of a lattice \(L\) if there is a retraction \(r\), of \(L\) onto \(M\), and \(s\) is then called a pseudo-inverse of \(r\).

**Definition 1.3** Let \(L\) and \(M\) be bounded lattices. We say that \(M\) is a sublattice of \(L\) if \(M\) is a retract of \(L\). In other words, \(M\) is a sublattice of \(L\) if there is a retraction \(r\) of \(L\) onto \(M\).

The purpose of defining sublattices as done above is to provide a relaxed notion of this concept in which the classical requirement that \(M\) must be a subset of \(L\) is disregarded. Generally, given a bounded lattice \(L\) it is said that a set \(M\) is a (usual) sublattice of \(L\) if it satisfies the following conditions:

- SR1 - \(M\) is a subset of \(L\);
• SR2 - M endowed with the restriction of the operations in L is also a lattice.

Notice that in Definition 1.3 the property SR1 is not required while property SR2 is somewhat redefined. Actually, an identification of M with a subset K = s(M) of L is done in order to carry on some properties of M to K, including its lattice structure via retraction r (see Figure 1). In this case, K works as an algebraic copy of M embedded into L since r is a homomorphism.

**Remark 1.2** Throughout this paper, it is used the concept of sublattice as in Definition 1.3. Whenever the usual definition of sublattice is used and this is not clear from the context, this sublattice will be called ordinary sublattice.

The main advantage behind the idea of using this relaxed version of sublattice is that it allows us to try to verify the validity for L of a property which is invariant under homomorphisms from a lattice M without requiring that M is a subset of L.

**Definition 1.4** Every retraction \( r : L \rightarrow M \) (with pseudo-inverse s) which satisfies \( s \circ r \leq id_L \) \(^1\) \((id_L \leq s \circ r)\) is called a lower (an upper) retraction. In this case, M is a lower (an upper) retract of L.

**Example 1.1** Let M and L be bounded lattices as shown in Figure 2. A mapping \( r : L \rightarrow M \) given by \( r(x) = \sup_M \{ z \in M \mid s(z) \leq_L x \} \) is a lower retraction whose pseudo-inverse is the mapping \( s : M \rightarrow L \) defined by \( s(1_M) = 1_L \), \( s(a) = v \), \( s(b) = x \), \( s(c) = y \), \( s(d) = z \) and \( s(0_M) = 0_L \). Therefore, it follows that M is a sublattice of L in the sense of Definition 1.3.

**Remark 1.3** Note that, given a lower retraction it is possible sometimes to define an upper retraction with the same pseudo-inverse. For instance, let L and M be lattices as shown in

\(^1\)If \( f \) and \( g \) are functions on a lattice \( L \) it is said that \( f \leq g \) if and only if \( f(x) \leq_L g(x) \) for all \( x \in L \).
Figure 2. If \( r \) is a lower retraction with pseudo-inverse \( s \) as defined in the Example 1.1, then the function \( r' \) given by \( r'(x) = \inf_M \{ z \in M \mid s(z) \geq_L x \} \) is an upper retraction since \( \text{id}_L \leq s \circ r' \), and it is easy to check that its pseudo-inverse is also \( s \).

It is worth noting that if \( M \) is a sublattice of \( L \) then there is a retraction \( r \) from \( L \) onto \( M \) but it is not required to \( r \) to be a lower or an upper retraction. Nevertheless, as shown in the above remark, there may be more than one retraction from \( L \) onto \( M \) with the same pseudo-inverse. This is a very useful particularity of Definition 1.3 and we would like to highlight it in a definition.

2 T-norms and Negations on \( L \)

Classical conjunctions have been modeled in fuzzy logics via functions called t-norms [6, 14, 15]. In this subsection we present the natural generalization to consider arbitrary bounded lattices as possible truth values.

**Definition 2.1** A mapping \( T : L \times L \to L \) is a t-norm on \( L \) if the following properties are satisfied for each \( x, y, z \in L \):

(T1) \( T(x, y) = T(y, x) \);

(T2) \( T(x, T(y, z)) = T(T(x, y), z) \);

(T3) if \( y \leq_L z \) then \( T(x, y) \leq_L T(x, z) \); and

(T4) \( T(x, 1_L) = x \)

\( T \) is positive if for each \( x, y \in L \) it satisfies the property

(T5) \( T(x, y) = 0_L \) iff \( x = 0_L \) or \( y = 0_L \).

An \( x \in L \) is an idempotent element of \( T \) if \( T(x, x) = x \).

Fuzzy negations are generalizations of the classical negation \( \neg \) and, as in classical logics, they have been used to define other connectives from binary connectives.

**Definition 2.2** A mapping \( N : L \to L \) is a negation on \( L \) or just an \( L \)-negation, if the following properties are satisfied for each \( x, y \in L \):

(N1) \( N(0_L) = 1_L \) and \( N(1_L) = 0_L \) and

(N2) If \( x \leq_L y \) then \( N(y) \leq_L N(x) \).

Moreover, the \( L \)-negation \( N \) is strong if it also satisfies the involutive property, i.e.

(N3) \( N(N(x)) = x \) for each \( x \in L \).

The \( L \)-negation \( N \) is called frontier if it satisfies the property:

(N4) \( N(x) \in \{0_L, 1_L\} \) iff \( x = 0_L \) or \( x = 1_L \).

Observe that the property (N4) is the analogous to the positive t-norm and each strong \( L \)-negation is a frontier \( L \)-negation.
3 Negation on \( L \) obtained from t-norms on \( L \)

In [8, 9] it was observed that it is possible to obtain, in a canonical way, a fuzzy negation \( N_T \) from a t-conorm \( T \). This negation is called natural negation of \( T \) or negation induced by \( T \). In the most general case, where we have a t-conorm on a bounded lattice \( L \), it is not always possible to obtain a fuzzy negation, because the construction of \( N_T \) is based in the supremum of, possibly, infinite sets.

**Proposition 3.1** Let \( L \) be a complete lattice and \( T \) be a t-norm on \( L \). Then the function \( N_T : L \to L \) defined by

\[
N_T(x) = \sup\{z \in L : T(x, z) = 0_L\}
\]

is an \( L \)-negation.

**Proof:**

(N1) \( N_T(1_L) = \sup\{z \in L : T(1_L, z) = 0_L\} = \sup\{0_L\} = 0_L \) and \( N_T(0_L) = \sup\{z \in L : T(0_L, z) = 1_L\} = \sup L = 1_L \).

(N2) If \( x \leq y \) then for any \( z \in L \), \( T(x, z) \leq T(y, z) \) and therefore, if \( T(y, z) = 0_L \) then \( T(x, z) = 0_L \). So, \( \{z \in L : T(y, z) = 0_L\} \subseteq \{z \in L : T(x, z) = 0_L\} \). Hence, \( N_T(y) = \sup\{z \in L : T(y, z) = 0_L\} \leq \sup\{z \in L : T(x, z) = 0_L\} = N_T(x) \).

**Theorem 3.1** Let \( T \) be a t-norm on \( L \). If \( T \) is positive then \( N_T = N_\perp \).

**Proof:** If \( x \neq 0_L \) and \( z \in L \) then, by (T5), \( T(x, z) = 0_L \) iff \( z = 0_L \). So, by eq. (1), \( N_T(x) = \sup\{0_L\} = 0_L \). Therefore, \( N_T = N_\perp \).

**Theorem 3.2** Let \( T \) be a t-norm on \( L \). If \( N_T \) is a frontier negation then each \( x \in L - \{0_L\} \) is a zero divisor of \( T \).

**Proof:** If \( x \neq 1_L \), then, as \( N_T \) is frontier, \( N_T(x) \neq 0_L \) and so \( \{z \in L : T(x, z) = 0_L\} \neq 0_L \). So, \( \{z \in L : T(x, z) = 0_L\} \neq \{0_L\} \). Thus, since \( T(x, 0_L) = 0_L \), \( \{0_L\} \subset \sup\{z \in L : T(x, z) = 0_L\} \). Therefore, there exists \( z \in L - \{0_L\} \) such that \( T(x, z) = 0_L \). Hence, \( x \) is a zero divisor of \( T \).

**Theorem 3.3** Let \( T \) be a t-norm on \( L \) and \( \rho \) be an automorphism on \( L \). Then \( N_T^\rho = N_T \rho \).

**Proof:** Let \( x \in L \), then \( N_T^\rho(x) = \rho^{-1}(N_T(\rho(x))) = \rho^{-1}(\sup\{z \in L : T(\rho(x), z) = 0_L\}) = \sup\{\rho^{-1}(z) \in L : T^\rho(\rho(x), \rho^{-1}(z)) = 0_L\} = \sup\{z \in L : T^\rho(x, z) = 0_L\} = N_T(\rho(x)) \).

4 Extension of T-norms and Negations

Let \( M \) be a sublattice of lattice \( L \) and \( T \) a t-norm on \( M \). In particular, \( T \) is a function, which brings forth the interesting question of whether it is possible to extend it to a t-norm \( T^E \) on \( L \) such that \( T^E \) acts on \( s(M) \) as a t-norm, \( T \), on \( M \), or not. In other words, how can we determine an extension of \( T \) to \( L \) in order to preserve the t-norm properties?

In general, this is not the case. In fact, as shall be shown in this section, it is necessary to impose some special properties. The problem of extending the functions has been well studied question in a number of mathematical theories. For instance, given a continuous map, \( f : W \to Y \), where \( W \) is a subset of a topological space \( X \), it is not always true that \( f \) might be extended to \( F : X \to Y \) preserving continuity, and such that \( F|_W = f \), unless \( W \) is a closed set [10].

Let \( M \) be a sublattice of bounded lattice \( L \) in the sense of definition 1.3 and suppose there is a retraction, \( r : L \to M \), and a pseudo-inverse, \( s : M \to L \), such that \( r \circ s = Id_M \). Thus, if \( T \) is a t-norm on \( M \), we would like to provide a t-norm \( T^E \) on \( L \) such that \( T^E|_{s(M) \times s(M)} = T \circ (r \times r) \). However, this is not the case even when \( M \) is a sublattice in the usual sense.

A good solution for this problem was presented in [11], as we can see on the next theorem.
**Theorem 4.1** Let $L$ be a bounded lattice and $M$ be a complete sublattice of $L$. If $T$ is a t-norm on $M$ and $r : L \rightarrow M$ is a lower retraction with pseudo-inverse $s : M \rightarrow L$ then, $T^E : L \times L \rightarrow L$, defined by

$$T^E(x, y) = \begin{cases} x \land_L y, & \text{if } 1_L \in \{x, y\} \\ s(T(r(x), r(y))), & \text{otherwise.} \end{cases}$$

(2)

is a t-norm that extends $T$ from $M$ to $L$.

In a similar way it is possible to extend negations.

**Proposition 4.1** Let $L$ be a bounded lattice, $M$ a complete sublattice of $L$ and $N : M \rightarrow M$ a fuzzy negation. If $r : L \rightarrow M$ is a retraction, the pseudo-inverse of which is $s : M \rightarrow L$, then $N^E(x) = s(N(r(x)))$ for each $x \in L$ is a fuzzy negation that extends $N$ from $M$ to $L$.

**Proof:** See [11].

5 **Extension of $N_T$**

Given a t-norm $T$ on a complete sublattice $M$ of complete lattice $L$ it shown in this section the relationship between $N_{TE}$ and $(N_T)^E$.

**Theorem 5.1** Let $M$ be a complete sublattice of $L$. Suppose that $r_1, r_2 : L \rightarrow M$ are a lower and upper retraction respectively with the same pseudo-inverse $s : M \rightarrow L$ such that $s \circ r_1 \leq id_L \leq s \circ r_2$. Moreover, suppose that $r_1(x) = 0_M$ iff $x = 0_L$. If $T$ is a t-norm on $M$, then $(N_T)^E \leq N_{TE}$.

**Proof:** By Theorem 2 and Proposition 3.1 it follows that

$$N_{TE}(x) = \sup\{y \in L \mid T^E(x, y) = 0_L\}$$

(3)

and

$$(N_T)^E(x) = s(N_T(r_1(x))) = s(\sup\{z \in M \mid T(r_1(x), z) = 0_M\})$$

(4)

Take an arbitrary $x \in L$ and let be $A = \{y \in L \mid T^E(x, y) = 0_L\}$. We will prove that $(N_T)^E(x) \in A$ and hence it is possible to conclude that $(N_T)^E(x) \leq N_{TE}(x)$ since $N_{TE}(x) = \sup A$.

Thus, we shall prove that $T^E(x, (N_T)^E(x)) = 0_L$. In fact, if $x = 1_L$ it follows that $(N_T)^E(x) = 0_L$ then by Theorem 2 we have $T^E(x, (N_T)^E(x)) = 0_L$. On the other hand, if $(N_T)^E(x) = 1_L$ then $1_L = s(N_T(r_1(x))) = s(\sup\{z \in M \mid T(r_1(x), z) = 0_M\})$ and hence $\sup\{z \in M \mid T(r_1(x), z) = 0_M\} = 1_M$ since $s$ is injective. Thus, $\{z \in M \mid T(r_1(x), z) = 0_M\} = M$ that implies $r_1(x) = 0_M$, i.e. $x = 0_L$. Then it can be concluded that $T^E(x, (N_T)^E(x)) = 0_L$.

Otherwise, when $1_L \notin \{x, y\}$ we have that

$$T^E(x, (N_T)^E(x)) = s(T(r_1(x), r_1((N_T)^E(x))))$$

$$= s(T(r_1(x), r_1(s(N_T(r_1(x))))))$$

$$= s(T(r_1(x), N_T(r_1(x))))$$

(5)

Note that $N_T(r_1(x)) = \sup\{z \in M \mid T(r_1(x), z) = 0_M\}$ then $T(r_1(x), N_T(r_1(x))) = 0_M$. Thus $s(T(r_1(x), N_T(r_1(x)))) = 0_L$ and hence $T^E(x, (N_T)^E(x)) = 0_L$ by (5).

Therefore, $(N_T)^E(x) \leq N_{TE}(x)$ for all $x \in L$, i.e. $(N_T)^E \leq N_{TE}$.
6 Final Remarks

It was discussed in this paper some results relating a particular class of negations and the problem of extending fuzzy operators. The main result, Theorem shown that given a t-norm $T$ on $M$ no matter if you extend $T$ and then take $N_{T^E}$ or if you take $N_T$ and then extend it, the resulting negation is the same.

Further, we would like to verify a similar result for the extension of negation $N_T^p$ (see Theorem 3.3) and other questions as can be seen on [11].

Referências


