TEMA Tend. Mat. Apl. Comput., **10**, No. 1 (2009), 99-110.

Static Output Feedback Stabilization using Invariant Subspaces and Sylvester Equations

E.R.L. VILLAREAL[1], Departamento de Ciências Exatas e Naturais, Universidade Federal Rural do Semi-árido DCEN/UFERSA, BR 110, Km 47, Bairro Costa e Silva, 59625-900 Mossoró, RN, Brasil.

J.A. RUIZ VARGAS[2], E.M. HEMERLY[3], Instituto Tecnológico de Aeronáutica, Praça Mal. Eduardo Gomes, 50, ITA-IEE-IEES Vila das Acácias, 12228-900 São Jose dos Campos, SP, Brasil.

Abstract. This paper presents systematic computational algorithms for obtaining the output feedback gain matrix in linear systems stabilization problems. Based on the concept of (C, A, B)-invariant subspaces, introduced previously by the first author, that has related the existence of a gain matrix to the solution of coupled Sylvester equations, two algorithms are presented: 1) in the Syrmos-Lewis algorithm, a modification is proposed to provide a more adequate framework to numerical solution, and 2) by using orthogonal transformations, the Alexandridis-Paraskevopoulos algorithm is modified to overcome, in part, the Kimura condition. Numerical examples are provided to illustrate the application of the proposed algorithms.

Keywords. Stabilization, Output Feedback, Invariant Subspace, Sylvester equations.

1. Introduction

It is well known that the static output feedback stabilization problem is a relevant and challenging problem, due to its simple implementation, in contrast to more sophisticated dynamic methods, and because it is not easy to find a feedback matrix to make the closed-loop system asymptotically stable [5], [15], [10], [8], [2], unlike to the state feedback case.

The determination of procedures for obtaining the feedback matrix in output feedback control to asymptotically stabilize linear systems is actually an open problem. Several methods have been proposed in the literature based on Lyapunov, Ricatti, linear matrix inequalities, or eigenstructure assignement. See, for instance [1]-[5], [8], [10]-[14], [16], and the references therein.

[1]elmerllanos@ufersa.edu.br
[2]vargas@ieee.org
[3]hemerly@ita.br

In particular, recently in [5] it was introduced the concept of (C, A, B)-invariant subspaces to relate the existence of a feedback gain matrix to the solution of two coupled Sylvester equations. However, this solution was mainly obtained under the Kimura condition [11] (i.e., $m + p > n$, where n, m, and p are the dimensions of the state, input, and output, respectively). Pathological cases where $m + p \leq n$, were only tackled by a trial and error procedure [5].

Hence, motivated by the previous facts, the main concern in this paper is the design, under a less restrictive than Kimura condition, of more efficient computational algorithms to the selection of the output feedback matrix. Based on the Syrmos-Lewis algorithm, a modification is proposed to provide a more adequate framework to numerical solution. Basically, by considering that the system is observable, controllable and under the Kimura condition, the coupled Sylvester equations are solved by assuming that the dimension of $\mathcal{V} = Im\,(V)$ is equal to the number of outputs p and the matrix $CV \in \Re^{p\times p}$ is invertible. Then the feedback matrix can be found as $G = W(CV)^{-1}$, where V,T, and W are design matrices. In the sequel, to obtain the feedback matrix, the Alexandridis-Paraskevopoulos algorithm is modified. A simple algorithm is proposed under a less restrictive Kimura condition (it is now assumed that $n \leq mp$). It should be noted that in contrast to [5], the proposed algorithm uses a coordinate transformation and reduced order system to alleviate the computational burden. Both techniques are based on the subspace intersection condition, $Ker\,(T) \cap Ker\,(C) = \{0\}$, which was obtained in [5].

Considered the linear time-invariant system described by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.1}$$

$$y(t) = Cx(t) \tag{1.2}$$

where $x \in \mathcal{X} \sim \Re^n$, $u \in \mathcal{U} \sim \Re^m$, $y \in \mathcal{Y} \sim \Re^p$. It is assumed that B is full column-rank , C is full row-rank and that (C, A, B) is stabilizable and detectable.

The problem here is to find a static output feedback control law

$$u(t) = Gy(t) \quad , \quad G \in \Re^{m\times p} \tag{1.3}$$

such that $\sigma(A + BGC) \in \mathcal{C}^-$, or equivalently, the closed-loop system is asymptotically stable, where $\sigma(A + BGC) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the set of eigenvalues of $A + BGC$.

The reminder of the paper is organized in as follows. In Section 2, basic concepts on geometric control are introduced. Section 3 presents the main contributions of the paper: the proposed algorithms for obtaining the feedback gain matrix with examples. Finally, in Section 4, the conclusions of the paper are presented.

2. Invariant Subspaces and Coupled Sylvester Equations

In this Section some concepts and definitions on geometric control theory are introduced [17]. A subspace $\mathcal{V} \subset \mathcal{X}$ is (A, B)-invariant if there exists $K : \mathcal{X} \longrightarrow \mathcal{Y}$, such that $(A + BK)\mathcal{V} \subset \mathcal{V}$, or, equivalently , if $A\mathcal{V} \subset \mathcal{V} + Im\,B$. In a dual manner, a subspace $\mathcal{T} \subset \mathcal{X}$ is (C, A)-invariant if exist $F : \mathcal{Y} \longrightarrow \mathcal{X}$, such that $(A+FC)\mathcal{T} \subset \mathcal{T}$, or, equivalently, if $\mathcal{T} \supset A(\mathcal{T} \cap Ker\,(C))$.

Definition 2.1. *[14], [16] A subspace $\mathcal{V} \subset \mathcal{X}$, of dimension v , is (C, A, B)-invariant if $\mathcal{V}$ is (A, B)-invariant and (C, A)-invariant.*

Consider $V \in \Re^{v\times v}$, such that $Im\,(V) = \mathcal{V}$, and let $T \in \Re^{(n-v)\times v}$ be a left annihilator of V, $i.e.$, $Ker\,(T) = Im\,(V)$. The definition 2.1 is equivalent to the existence of matrices ($H_V \in \Re^{v\times v}$, $W \in \Re^{m\times v}$) and ($H_T \in \Re^{n-v\times n-v}$, $U \in \Re^{(n-v)\times v}$), which are solutions of the following *coupled Sylvester Equations*

$$AV - VH_V = -BW \tag{2.1}$$
$$TA - H_T T = -UC \tag{2.2}$$
$$TV = 0. \tag{2.3}$$

The definition 2.1 and the equations (2.1) , (2.2), and (2.3) have a crucial importance in static output feedback via eigenstructure assignment [12] [14].

The properties of stabilizability and detectability are considered, respectively, by using two definitions
(i) a subspace (A, B)-invariant $\mathcal{V}$ is (A, B)-inner stabilizable subspace, or, equivalently, there exists K such that $(A + BK)|\mathcal{V}$ is asymptotically stable ; and
(ii) a subspace (C, A)-invariant $\mathcal{V}$ is (C, A)-outer detectable subspace if there exist F such that $(A + FC)|\mathcal{X}/\mathcal{V}$ is asymptotically stable.

Definition 2.2. *[14] A subspace $\mathcal{V}$, of dimension v, is (C, A, B)-invariant output stabilizable subspace , (or simply O.S. (C, A, B)-invariant [4]) if $\mathcal{V}$ is (A, B)-inner stabilizable and (C, A)-outer detectable.*

Thus, a necessary and sufficient condition for the existence of $\mathcal{V} = Im(V)$ or $O.S.$ (C, A, B)-invariant subspace is that (2.1), (2.2) and (2.3) are verified with the additional condition of stability

$$\sigma(H_V) \in \mathcal{C}^- \tag{2.4}$$
$$\sigma(H_T) \in \mathcal{C}^- . \tag{2.5}$$

Theorem 2.1, in what follows, relates the concept of $O.S.$ (C, A, B)-invariant subspace to the existence of a static output feedback control law (1.3), which stabilizes the closed loop system.

Theorem 2.1. *[5] There exists a static output feedback matrix $G : \mathcal{Y} \longrightarrow \mathcal{U}$ such that $\sigma(A + BGC) \in \mathcal{C}^-$, if and only if the following conditions hold true for some matrices ($V \in \Re^{n\times v}, H_V \in \Re^{v\times v}, W \in \Re^{m\times v}$), ($T \in \Re^{n-v\times n}, H_T \in \Re^{n-v\times n-v}, U \in \Re^{n-v\times v}$), and for some positive integer $v \leq n$*

$$AV - VH_V = -BW, \textit{with } \sigma(H_V) \in \mathcal{C}^- \tag{2.6}$$
$$TA - H_T T = -UC, \textit{with } \sigma(H_T) \in \mathcal{C}^- \tag{2.7}$$
$$TV = 0 \tag{2.8}$$
$$Ker\,(CV) \subseteq Ker\,(W) \tag{2.9}$$
$$Ker\,(B'T') \subseteq Ker\,(U') \tag{2.10}$$

[4] Output Stabilizable (C,A,B)-invariant [14] [4]

where $rank\,(V) = v$ *and* $rank\,(T) = n - v$.

It is interesting to notice that these results have been presented and explored under different forms in the literature related to eigenstructure assignment by output feedback (see [15]). The presented statement corresponds to Theorem 3.2 in [14]. The *coupled Sylvester Equations* (2.6), (2.7), and (2.8) describe some geometric properties of the subspace $\mathcal{V} = Im\,(V)$. Under stability of the matrix H_V, the equation (2.6) means that the subspace $\mathcal{V} = Im\,(V)$ must be (A, B)-inner stabilizable. In a dual manner, the subspace $Ker\,(T) = \mathcal{V}$ must be (C, A)-outer detectable, see (2.7). Thus, the coupling condition (2.8), the existence of $\mathcal{V} = Ker\,(T)$ *O.S.* (C, A, B)-invariant subspace, and the conditions (2.9) and (2.10), are necessary and sufficient conditions for the existence of a matrix G.

Also, note that the conditions (2.9) and (2.10) correspond to the existence of a matrix $G \in \Re^{m\times p}$, which verifies the following two equalities

$$GCV = W \tag{2.11}$$

$$TBG = U, \tag{2.12}$$

where the closed loop system has eigenvalues given by

$$\sigma(A + BGC) = \sigma(H_V)\,\dot{\cup}\,\sigma(H_T).$$

3. Algorithmic Aspects

In this section, based on a subspace intersection condition, i.e., $Ker\,(T) \cap Ker\,(C) = \{0\}$ [5], two algorithms to eigenstructure assignment are presented. Motivated by the algorithm of Syrmos and Lewis [12], the first technique solves the coupled Sylvester equations in two steps, and can directly be applied to systems that verify the Kimura condition [11]. The second technique is based on the algorithm proposed by Alexandridis and Paraskevopoulos [1]. It uses a system representation in an adequate base of the state space to rewrite the Sylvester equations in a set of bilinear algebraic equations. The Proposed techniques assume that the dimension p of the output is the same as $\mathcal{V} = Im\,(V)$. Also, it is guaranteed that the matrix $CV \in \Re^{p\times p}$ is invertible, so that $G = W(CV)^{-1}$, see (2.11).

The set of the desired closed loop poles is denoted by $\Lambda = \{\lambda_1, \ldots, \lambda_{n-p}, \lambda_{n-p+1}, \ldots, \lambda_n\} = \{\Lambda_T, \Lambda_V\}$, where Λ_T and Λ_V are autoconjugated. $H_V \in \Re^{p\times p}$ and $H_T \in \Re^{(n-p)\times(n-p)}$ are defined as $\sigma(H_T) = \Lambda_T = \{\lambda_1, \ldots, \lambda_{n-p}\}$ and $\sigma(H_V) = \Lambda_V = \{\lambda_{n-p+1}, \ldots, \lambda_n\}$. For simplicity, it is considered that the elements of $\Lambda = \sigma(A + BGC)$ are all distinct.

3.1. Modified algorithm of Syrmos and Lewis

Based on Theorem 2.1, the following algorithm, considered in [12], can be used to obtain an output feedback matrix that stabilizes the closed loop system, when the Kimura condition, $n < m + p$, is verified.

Step 1: Select the matrix $H_T \in \Re^{n-p\times n-p}$, where $\sigma(H_T) = \Lambda_T \in \mathcal{C}^-$, and solve (2.7) to find the matrix $T \in \Re^{n-p\times n}$, such that

$$rank\left(\begin{bmatrix} T \\ C \end{bmatrix}\right) = n \Longleftrightarrow Ker\ T \cap Ker\ C = \{0\}. \tag{3.1}$$

Step 2.1: Solve (2.6) for some matrix $H_V \in \Re^{p\times p}$, where $\sigma(H_V) = \Lambda_V \in \mathcal{C}^-$, whereas the matrix V satisfy the coupling condition (2.8) and $rank\,(V)$ is equal to p.
Step 2.2: By construction, the relation (3.1) guarantees that rank $(CV) = p$. Then G can be calculated as unique solution of (2.11).

Remark 3.1. *Steps* 1 *and* 2 *can be implemented by using standard techniques for eigenstructure assignment. By considering H_T e H_V in the Jordan form, the following procedure can be adopted:*

Step 1: *Find $t_j \in \mathcal{C}^n$ and $u_j \in \mathcal{C}^p$, such that*

$$\begin{bmatrix} t'_j & u'_j \end{bmatrix}\begin{bmatrix} A-\lambda_j I \\ C \end{bmatrix} = 0\ \forall\ j = 1,\ldots,n-p \tag{3.2}$$

The row matrix $T \in \Re^{(n-p)\times n}$, denoted for T_j, is constructed by using the vectors t_j, as follows

- *if $\lambda_j \in \Re$, then $T_j = t'_j$;*
- *if $\lambda_j \in \mathcal{C}$, then $\lambda_{j+1} = \lambda_j^*$ and* $\left\{\begin{array}{c} T_j = Re\,(t'_j) \\ T_{j+1} = Imag\,(t'_j) \end{array}\right.$.

Step 2: *To determine $v_i \in \mathcal{C}^n$ and $w_i \in \mathcal{C}^n$ we have*

$$\begin{bmatrix} A-\lambda_i I & B \\ T & 0 \end{bmatrix}\begin{bmatrix} v_i \\ w_i \end{bmatrix} = 0\ \forall\ i = n-p+1,\ldots,n. \tag{3.3}$$

In similar form to the previous case, the used matrices V and W can only be built with real elements. In particular, if $\lambda_i \in \mathcal{C}$, then $\lambda_{i+1} = \lambda_i^*$ and $\left\{\begin{array}{ll} V_i = Re\,(v_i), & V_{i+1} = Imag\,(v_i) \\ W_i = Re\,(w_i), & W_{i+1} = Imag\,(w_i) \end{array}\right.$ *, where v_i and w_i denote the columns of the matrices V and W, respectively.*

Remark 3.2. *[5] Step* 1*, in Remark 3.1, is concerned with the choice of a matrix T, which verifies (3.2). Step* 2 *is concerned with the solution of (2.6) and (2.8), to obtain the (A,B)-invariant of $Ker\,(T)$. This property is associated to the "zero equation" (3.3) [3]. To guarantee the freely placement of the eigenvalues λ_i in step* 2 *, the system matrix* $P(\lambda) = \begin{bmatrix} A-\lambda I & B \\ T & 0 \end{bmatrix}$ *of dimension $(2n-p)\times(n+m)$, must be row-full rank $\forall\lambda$. The algorithm is based in the fact that under Kimura condition, $m+p>n$, $P(\lambda)$ does not lose rank for almost all triples (A,B,T) [13]. Thus the zero equation (3.3) has solutions for all λ_i. In the presence of invariant zeros, it is possible to remake Step* 1 *to search other solutions for the Sylvester equation (2.6).*

Remark 3.3. *In the less restrictive case $m+p=n$, $P(\lambda)$ is square of dimension $(2n-p)\times(2n-p)$, and all triple (A,B,T) has p finite invariant zeros [12]. In this case, the basic procedure can produce stabilizing solutions only if T, found in step 1, generates p stable invariant zeros, which have to be used to solve (3.3) [3].*

Example 3.1. *Consider the linear system defined by [9]:*

$$A=\begin{bmatrix} 0 & 1 & 0 & 0\\ 1 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix};\ B=\begin{bmatrix} 0 & 0\\ 1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix};\ C=\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

The system is controllable and observable and $m+p=5>n$. The desired eigenvalues are $\Lambda_T=\{-4\}\cup\Lambda_V=\{-3,-2,-1\}$.

Step 1 : *By considering that $\lambda_1=-4$, and using (3.2) we have*
$T=\begin{bmatrix} -0.3148 & 0.0630 & -0.2406 & -0.2425\end{bmatrix}$, *where (A,B,T) has not invariant zeros*

Step 2.1 : *For $\lambda_2=-3$, $\lambda_3=-2$ and $\lambda_4=-1$, by using (3.3) we have*

$$V=\begin{bmatrix} 0.0727 & 0.1324 & 0.2425\\ -0.2182 & -0.2649 & -0.2425\\ 0.0242 & -0.0662 & 0.2225\\ -0.1751 & -0.3064 & -0.6184\end{bmatrix},\ W=\begin{bmatrix} 0.7999 & 0.6622 & -0.2424\\ 0.5254 & 0.6128 & -0.6184\end{bmatrix}.$$

Step 2.2 : *Based on $GCV=W$, we have*
$G=\begin{bmatrix} -46.9778 & -20.7333 & -26.9444\\ -16.1410 & -9.3519 & -10.9968\end{bmatrix}$. *The closed loop matrix $A_G=A+BGC$ is then*

$$A_G=\begin{bmatrix} 0.0000 & 1.0000 & 0.0000 & 0.0000\\ -45.97778 & 1 & -20.7333 & -26.9444\\ -1.0000 & 0.0000 & 0.0000 & 0.0000\\ -16.1410 & 0.0000 & -9.3519 & -10.9968\end{bmatrix}.$$

3.2. Modified algorithm of Alexandridis and Paraskevopoulos

The presented technique is based on the use of coordinate transformations, obtained from the matrix C, that allows the solution of equation (3.2) by using a auxiliary system of reduced order $n-p$.

This type of decomposition is used in control theory, mainly for the construction of minimum order observers [1], [6]. From a geometric point of view, it aims at the construction of a subspace $\mathcal{V}=Ker\,(T)$ outer stabilizable, and to guarantee $Ker\,(T)\cap Ker\,(C)=\{0\}$.

Consider a base change given by

$$x=\begin{bmatrix} P_1 & P_2\end{bmatrix}\begin{bmatrix} x_1\\ x_2\end{bmatrix}, \qquad (3.4)$$
$$P_1\in\Re^{n\times p},\quad P_2\in\Re^{n\times n-p},$$

where $P=\begin{bmatrix} P_1 & P_2\end{bmatrix}$ is nonsingular such that

$$C\begin{bmatrix} P_1 & P_2\end{bmatrix}=\begin{bmatrix} C_1 & 0\end{bmatrix}, \qquad (3.5)$$
$$\text{with}\ \ C_1\in\Re^{p\times p}\ \ \text{and}\ \ rank(C_1)=p.$$

The inverse of P is denoted for $\bar{P} = \begin{bmatrix} \bar{P}_1' \\ \bar{P}_2' \end{bmatrix}$, where $\bar{P}_1 \in \Re^{n \times p}$, $\bar{P}_2 \in \Re^{n \times n-p}$, and $\begin{bmatrix} P_1 & P_2 \end{bmatrix} \begin{bmatrix} \bar{P}_1' \\ \bar{P}_2' \end{bmatrix} = I_n$.

In this base, the open loop system can be rewritten as

$$\begin{aligned} \begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{aligned} \tag{3.6}$$

where, in particular, $A_{12} \in \Re^{p \times n-p}$ and $A_{22} \in \Re^{n-p \times n-p}$.

Lemma 3.1. *If (C, A) is observable, then there exists a $T \in \Re^{n-p \times p}$ such that (2.7) is verified and $Ker\,(T) \cap Ker\,(C) = \{0\}$.*

Proof. As shown in [6], the observability of the pair (C, A) is equivalent to the observability of the pair (A_{12}, A_{22}). $T_2 \in \Re^{(n-p)\times(n-p)}$, with $rank\,(T_2) = n - p$, and $T_1 \in \Re^{(n-p)\times p}$, are solutions of the Sylvester equations of *order-reduced*

$$T_2 A_{22} - H_T T_2 \quad = \quad -T_1 A_{12} \quad , \quad \sigma(H_T) \in \mathcal{C}^- \tag{3.7}$$

where $H_T \in \Re^{n-p \times n-p}$ is chosen such that $\sigma(H_T) \in \mathcal{C}^-$. Note that in general it is possible to find T_2 invertible when H_T does not contain the eigenvalues from matrix A_{22} [1]. Then, calculate $U \in \Re^{n-p \times p}$ as unique solution of

$$UC_1 = -\left(T_1 A_{11} + T_2 A_{21} - H_T T_1\right). \tag{3.8}$$

Jointly, (3.7) and (3.8) verify

$$\begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} - H_T \begin{bmatrix} T_1 & T_2 \end{bmatrix} = U \begin{bmatrix} C_1 & 0 \end{bmatrix}$$

which, in the original base, corresponds to Sylvester equation (2.7). Moreover, as T_2 is invertible, we have

$$rank\left(\begin{bmatrix} T \\ C \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ C_1 & 0 \end{bmatrix} \begin{bmatrix} \bar{P}_1' \\ \bar{P}_2' \end{bmatrix} \right) = n \Longleftrightarrow Ker\,(T) \cap Ker\,(C) = \{0\}. \qquad \square$$

Equations (2.7) and (3.7) are related to the construction of minimum order observers [6] [1]. Consider the eigenvalues of $(A_{22} + L_2 A_{12})$, where $L_2 \in \Re^{n-p \times p}$ is such that

$$T_2 L_2 \quad = \quad T_1. \tag{3.9}$$

Then, substituting (3.9) into (3.7) it follows that

$$T_2(A_{22} + L_2 A_{12}) \quad = \quad H_T T_2 \tag{3.10}$$

or

$$\sigma(H_T) = \sigma(A_{22} + L_2 A_{12}) \in \mathcal{C},$$

Note that the decomposition of P is not unique. In particular, it can computationally be obtained through ortogonal transformations, $\bar{P} = P'$ [1] [4]. In what follows, in Theorem 3.1, a better alternative than the introduced by Alexandridis-Paraskevopoulos in [1] is proposed to alliviate the computational burden.

Theorem 3.1. *Given the system (1.1), (1.2). Consider a decomposition in the form (3.6). Then, the matrix G verifies* $\sigma(A + BGC) = \sigma(H_V) \dot{\cup} \sigma(H_T)$, *if and only if,*

$$\begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} \bar{P}_1' \\ \bar{P}_2' \end{bmatrix} V = 0, \tag{3.11}$$

where ($W \in \Re^{m\times p}, V \in \Re^{n\times p}$) *and* ($T_1 \in \Re^{n-p\times p}, T_2 \in \Re^{n-p\times n-p}$) *satisfy the Sylvester equation (2.6) and (3.7),respectively, with* $det\,(CV) \neq 0$.

Proof. Necessity: It is considered that there exists an output feedback matrix $G \in \Re^{m\times v}$, so that the closed loop system is asymptotically stable, i.e., the Sylvester equations are verified for some positive $v \leq n$ and matrices T and V, such that $TV = 0$. More precisely,

$$\begin{aligned} AV - VH_V &= -BW\ , \quad \text{with } \sigma(H_V) \in \mathcal{C}^- \\ TA - H_T T &= -UC\ , \quad \text{with } \sigma(H_T) \in \mathcal{C}^- \\ TV &= 0 \\ Ker\,(CV) &\subseteq Ker\,(W) \\ Ker\,(B'T') &\subseteq Ker\,(U') \end{aligned}$$

where $rank\,(V) = v$ and $rank\,(T) = n - v$.

Under observability (detectability) condition, it is possible to find a matrix T. Once the matriz T is constructed, Step 2 allows the solution of (2.6) and (2.8), i.e., the (A,B)-invariant of $Ker\,(T)$.

Consider

$$x = \begin{bmatrix} P_1 & P_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad P_1 \in \Re^{n\times p}, \quad P_2 \in \Re^{n\times n-p}, \tag{3.12}$$

where $P = \begin{bmatrix} P_1 & P_2 \end{bmatrix}$ is a nonsingular matrix so that

$$C \begin{bmatrix} P_1 & P_2 \end{bmatrix} = \begin{bmatrix} C_1 & 0 \end{bmatrix}\ , \quad \text{with } C_1 \in \Re^{p\times p} \text{ and } rank(C_1) = p. \tag{3.13}$$

The inverse of P is denoted by $\bar{P} = \begin{bmatrix} \bar{P}_1' \\ \bar{P}_2' \end{bmatrix}$. Based on (3.6) and Lemma 3.1, it is possible to conclude that the observability of (C, A) is equivalent to the observability of (A_{12}, A_{22}). With change of base (3.4), we obtain T_1 e T_2 as

$$T_2 A_{22} - H_T T_2 = -T_1 A_{12}\ , \quad \sigma(H_T) \in \mathcal{C}^-\ .$$

Hence, $\left(\begin{bmatrix} TV \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} \bar{P}_1' \\ \bar{P}_2' \end{bmatrix} V \right) = 0$. Consequently,

$$\begin{aligned} AV - VH_V &= -BW\ , \quad \text{with } \sigma(H_V) \in \mathcal{C}^- \\ T_2 A_{22} - H_T T_2 &= -T_1 A_{12}\ , \quad \sigma(H_T) \in \mathcal{C}^- \\ TV &= 0. \end{aligned}$$

Sufficiency: Consider

$$\begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} \bar{P}_1' \\ \bar{P}_2' \end{bmatrix} V \quad = \quad 0 \tag{3.14}$$

Then, the matrices ($W \in \Re^{m\times p}, V \in \Re^{n\times p}$) and ($T_1 \in \Re^{n-p\times p}, T_2 \in \Re^{n-p\times n-p}$) satisfy (2.6) and (3.7), respectively, where $det\,(CV) \neq 0$.

The matrix $H_T \in \Re^{n-p\times n-p}$ is choosed so that $\sigma(H_T) = \Lambda_T \in \mathcal{C}^-$ and (2.7) is resolved to find $T \in \Re^{n-p\times n}$ such that

$$rank\left(\begin{bmatrix} T \\ C \end{bmatrix}\right) = n \Longleftrightarrow Ker\ T \cap Ker\ C = \{0\} \tag{3.15}$$

Equation (2.6) is solved, for some matrix $H_V \in \Re^{p\times p}$, so that $\sigma(H_V) = \Lambda_V \in \mathcal{C}^-$ and by considering that V verifies (2.8) and $rank\,(V)$ must be p. By construction, (3.1) ensures that rank$\,(CV) = p$ and G can be computed as the unique solution of (2.11). □

From the formulation presented in Theorem 3.1, the following procedure is proposed in [1] to poles assignment, when $mp \geq n$.

3.2.1. Procedure for the determination of output feedback matrix

Considering that H_V and H_T are in Jordan form, from (2.6) W and V can be written as

$$W = \begin{bmatrix} w_1, & w_2, & \ldots, & w_p \end{bmatrix}; \tag{3.16}$$

$$V = \begin{bmatrix} X^1 w_1, & X^2 w_2, & \ldots, & X^p w_p \end{bmatrix}. \tag{3.17}$$

where

$$X^i = (\lambda_i I - A)^{-1} B, \quad for \quad i = 1, \ldots, p. \tag{3.18}$$

Similarly, from (3.7) T_1 and T_2 can be written as

$$T_1 = \begin{bmatrix} \phi'_{p+1} \\ \phi'_{p+2} \\ \vdots \\ \phi'_n \end{bmatrix} \quad and \quad T_2 = \begin{bmatrix} \psi'_{p+1} \\ \psi'_{p+2} \\ \vdots \\ \psi'_n \end{bmatrix}$$

where ψ_j , $\forall j = p+1, \ldots, n$, can be obtain as

$$\psi'_j = \phi' A_{12}(\lambda_j I - A_{22})^{-1}. \tag{3.19}$$

From the notation above, the coupling equation (3.11) can be written in the form of a bilinear system of algebraic equations. Thus, substituting (3.18) and (3.19) in (3.11), we have

$$\phi'_j \begin{bmatrix} I_p & A_{12}[\lambda_j I - A_{22}]^{-1}\bar{P}_1' \\ \bar{P}_2' & \end{bmatrix} X^i w_i = 0 \tag{3.20}$$

$$for \quad i = 1, , \ldots, p \quad and \quad j = p+1, \ldots, n.$$

By solving (3.20), the parameters for w_i and ϕ'_j are determined. Then, the matrices W and V are defined based on (3.16), and the output feedback matrix G is directly calculated from (2.11). By using this formulation, the justification to the case $n \leq mp$ is given in [1]. "The equation (3.16) provides $p(m-1)$ free parameters (based on the arbitrary elements of $w_1, w_2, \ldots, w_p$). Additionally the equation (3.19) provides $(p-1)(n-p)$ free parameters (from the arbitrary elements of $\phi_{p+1}, \phi_{p+2}, \ldots, \phi_n$). On the other hand the equation (3.20) results in a system of $p(n-p)$ independent equations. Hence, for the solution existence, the number of equations must be equal or smaller than the addition of free parameters. After algebraic manipulations, the condition $mp \geq n$" is obtained .

Example 3.2. *Consider the previous example where* $C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.
The corresponding system is controllable and observable $mp = 4 = n$, $p = 2$ *and* $m = 2$. *The desired eigenvalues are* $\Lambda_V = \{-1, -2\}$ *and* $\Lambda_T = \{-3, -4\}$.

By using (3.6), a decomposition P *is founded i.e.,* $P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

such that $C \begin{bmatrix} P_1 & P_2 \end{bmatrix} = \begin{bmatrix} \bar{C}_1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. *Then* $\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} = \begin{bmatrix} \bar{P}'_1 \\ \bar{P}'_2 \end{bmatrix}$
$AP_2 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

For $\lambda_1 = -1, \lambda_2 = -2$, *two free parameters in* $\begin{bmatrix} w_1 & w_2 \end{bmatrix} = \begin{bmatrix} 1.0 & 1.0 \\ w_{12} & w_{22} \end{bmatrix}$ *are fixed,* $w_{12} = -2.2$ *and* $w_{22} = 2.8$. *On the other hand, for* $\lambda_3 = -3, \lambda_4 = -4$, *two free parameters in* $\begin{bmatrix} \phi'_3 \\ \phi'_4 \end{bmatrix} = \begin{bmatrix} 1.0000 & \phi_{32} \\ 1.0000 & \phi_{42} \end{bmatrix}$ *are fixed to obtain* $\phi_{32} = -1.6364$ *and* $\phi_{42} = -1.8824$. *By using the previous procedure, the matrices* (W, V) *and* (T_1, T_2, T), *which satisfy the Sylvester equations (2.6), (3.7) and (3.20), with* $det(CV) \neq 0$, *are determined as follows*

$$V = \begin{bmatrix} 0.6 & -0.8 \\ -1.6 & 0.6 \\ 1.6 & -1.2 \\ 0.6 & -0.4 \end{bmatrix}; \quad W = \begin{bmatrix} 1.0 & 1.0 \\ -2.2 & 2.8 \end{bmatrix};$$

$$T_1 = \begin{bmatrix} 1.0000 & -1.6364 \\ 1.0000 & -1.8824 \end{bmatrix}; \quad T_2 = \begin{bmatrix} -0.5455 & 0.1818 \\ -0.4706 & 0.1176 \end{bmatrix};$$

$$T = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} \bar{P}'_1 \\ \bar{P}'_2 \end{bmatrix} = \begin{bmatrix} -0.5455 & 0.1818 & 1.0000 & -1.6364 \\ -0.4706 & 0.1176 & 1.0000 & -1.8824 \end{bmatrix}.$$

The system (A, B, T) *has stable eigenvalues,* $\{-1, -2\}$, *as invariant zeros. The output feedback matrix* G *is determined by* $GCV = W$, *i.e.,* $G = \begin{bmatrix} -12.5 & 35.0 \\ -10.0 & 23.0 \end{bmatrix}$.

The closed loop matrix $A_G = A + BGC$*, whose eigenvalues are the desired ones, is given by* $A_G = \begin{bmatrix} 0.0 & 1.0 & -12.5 & 35.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & -10.0 & 24.0 \\ -1.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$.

4. Conclusion

In this paper two computational algorithms were proposed to obtain the output feedback matrix gain in stabilization problems of linear systems. In the first, two coupled Sylvester equations, based on the concept of $O.S.$ (C, A, B)-invariant subspace, were presented to compute the output feedback matrix. In the second, the Syrmos-Lewis and Alexandridis-Paraskevopoulos algorithms were modified by using the presented methodology to alleviate the computational burden and previous asumption on system structure.

Resumo. Este artigo apresenta em forma sistemática algoritmos computacionais para se obter a matriz de realimentação de saídas em problemas de estabilização de sistemas lineares. Baseado no conceito de (C, A, B)-subespaços invariantes, introduzido anteriormente pelo primeiro autor, e através do uso da solução de equações de Sylvester acopladas obtém-se a matriz de realimentação de saída. Dois algoritmos são apresentados: 1) modificação do algoritmo Syrmos-Lewis para fornecer um método mais adequado para solução numérica, e 2) utilizando transformações ortogonais modifica-se o algoritmo de Alexandridis-Paraskevopoulos com o objetivo de superar, em parte, a condição de Kimura. Exemplos numéricos são apresentados para ilustrar a aplicação dos algoritmos.

References

[1] A.T. Alexandridis, P.N. Paraskevopoulos, A new approach to eigenstructure assignment by output feedback, *IEEE Transactions on Automatic Control*, **41**, No. 7 (1996), 1046–1050.

[2] O. Bachelier, J. Bosche, D. Mehdi, On pole placement via eigenstructure assignment approach, *IEEE Transactions on Automatic Control*, **51**, No. 9 (2006), 1554–1558.

[3] E.B. Castelan, J.C. Hennet, Eigenstructure assignment for state constrained linear continuos time systems, *Automatica*, **28**, No. 3 (1993), 605–611.

[4] E.B. Castelan, J.C. Hennet, E.R.Ll. Villarreal, Quadratic characterization and use of output stabilizable subspaces, in "Proceedings (CD) of the 8th Mediterranean Conference on Control and Automation", Greece, 2000.

[5] E.B. Castelan, J.C. Hennet, E.R.Ll. Villarreal, Quadratic characterization and use of output stabilizable subspaces, *IEEE Transactions on Automatic Control*, **48**, No. 4 (2003), 654–660.

[6] C.T. Chen, "Linear System Theory ans Design", Holt, Rinehart and Winston, 1996.

[7] L. Dai, "Singular Control System", Springer-Verlag, 1989.

[8] J. Dong, G.-H. Yang,Static output feedback control synthesis for linear systems with time-invariant parametric uncertainties, *IEEE Transactions on Automatic Control*, **52**, No. 10 (2007), 1930–1936.

[9] L.R. Fletcher, J. Kautsky, G.K.G. Kolka, N.K. Nichols, Eigenstructure assingment by output feedback in descriptor systems, *IEEE Transactions on Automatic Control*, **42**, No. 4 (1985), 1457–1468.

[10] Y. He, Q.-G. Wang, An improved ILMI method for static output feedback control with application to multivariable PID control, *IEEE Transactions on Automatic Control*, **51**, No. 10 (2006), 1678–1683.

[11] H. Kimura, Pole assignment by gain ouptup feedback, *IEEE Transactions on Automatic Control*, **20**, No. 4 (1975) 509–516.

[12] V.L. Syrmos, F.L. Lewis, Output feedback eigenstructure assignment using two Sylvester equations, *IEEE Transactions on Automatic Control*, **38**, No. 3 (1993), 495–499.

[13] V.L. Syrmos, F.L. Lewis, Transmission zero assignment using descriptions, *IEEE Transactions on Automatic Control*, **38**, No. 7 (1993), 1115–1120.

[14] V.L. Syrmos, F.L. Lewis, Bilinear formulation for the output feedback problem in linear system, *IEEE Transactions on Automatic Control*, **39**, No. 2 (1994), 410–414.

[15] V.L. Syrmos, C.T. Abdallah, P. Dorato, K. Grigoriadis, Static output feedback - a survey, *Automatica*, **33**, No. 2 (1997), 125–137.

[16] E.R.Ll. Villarreal, J.D. Lima, Approach for stabilization by output feedback, invariant subspaces and Sylvester equations, in "Anais CNMAC 2008", Belém, PA, Brasil.

[17] W.M. Wonahm, "Linear Multivariable Control, a Geometric Approach", Springer-Verlag, 1979.