Compressible Aerodynamic Flow Simulations Using ENO and WENO Schemes in a Finite Volume Unstructured Grid Context

W.R. WOLF\textsuperscript{1}, Instituto Tecnológico de Aeronáutica, CTA/ITA, 12228-900 São José dos Campos, SP, Brasil.

J.L.F. AZEVEDO\textsuperscript{2}, Instituto de Aeronáutica e Espaço, CTA/IAE 12228-900 São José dos Campos, SP, Brasil.

Abstract. In this work the essentially non-oscillatory schemes (ENO) and the weighted essentially non-oscillatory schemes (WENO) are implemented in a cell centered finite volume context on unstructured grids. The 2-D Euler equations will be considered to represent the flows of interest. The ENO and WENO schemes have been developed with the purpose of accurately capturing discontinuities appearing in problems governed by hyperbolic conservation laws. In the high Mach number aerodynamic studies of interest, these discontinuities are mainly represented by shock waves. The entire reconstruction process of ENO and WENO schemes is described in detail for any order of accuracy with an emphasis to the implementation of the second-order accurate schemes. An agglomeration multigrid method is used to reach faster convergence to steady state. The solution of the transonic flow over a RAE2822 supercritical airfoil is presented in order to assess the capability implemented against data available in the literature.

1. Introduction

The motivation for the present work is the need for accurate simulations of high Mach number aerodynamic flows with strong discontinuities. In recent years several efforts have been made by the CFD group of Instituto de Aeronáutica e Espaço [3],[4] for the development of computational tools which are able to accurately capturing discontinuities such as the shock waves appearing in the aerodynamic flows of interest. Some upwind schemes such as the van Leer flux vector splitting [21], the Liou AUSM\textsuperscript{+} flux vector splitting [11] and the Roe flux difference splitting [16] were implemented and tested for second-order accuracy considering a TVD-MUSCL reconstruction [2] and Azevedo et al. [3] have shown that nominally second-order schemes presented results with an order of accuracy smaller than the expected in the solutions for unstructured grids. Besides this fact, it is well known that TVD

\textsuperscript{1}wolf@ita.br
\textsuperscript{2}azevedo@iae.cta.br
schemes have their order of accuracy reduced to first order in the presence of shocks due to the effect of limiters.

This observation initiated the development of essentially non-oscillatory (ENO) schemes introduced by Harten et al. [8] in which oscillations up to the order of the truncation error are allowed to overcome the drawbacks and limitations of TVD schemes. Subsequent on the development of ENO schemes, the weighted essentially non-oscillatory schemes were introduced by Liu et al. [12] with the purpose of present better convergence rate for stationary cases, better smoothing for the flux vectors and better accuracy using the same stencils than the ENO schemes.

In the present work the ENO and the WENO schemes are implemented in a cell centered finite volume context for unstructured grids with the 2-D Euler equations considered to represent the flows of interest. The ENO and WENO schemes have been developed with the purpose of accurately capturing discontinuities appearing in problems governed by hyperbolic conservation laws. In the high Mach number aerodynamic studies of interest in the present paper, these discontinuities are mainly represented by shock waves. The entire reconstruction process of ENO and WENO schemes is described in detail for any order of accuracy and the second-order scheme is implemented and tested.

For the ENO schemes, interpolation polynomials of one order less than the order of accuracy required in the solution are computed and these polynomials are a good approximation to the values of the conserved variables within the control volumes. These polynomials interpolate primitive variable values in Gauss quadrature points using stencils determined by a von Neumann neighborhood. The control volume moments and the mean values of primitive variables in the control volumes are used to compute the polynomial coefficients and, hence, one can compute the oscillation of the polynomials and select the smoothest among them through the values of these coefficients. While the ENO schemes use the smoothest polynomial, the WENO schemes use all ENO computed polynomials for the stencils and, therefore, they construct one polynomial only. Non negative weights, which must add up to one, are computed for every polynomial through oscillation indicators and the WENO polynomial is constructed by the sum of all ENO polynomials multiplied by the respective weights. The solution of the transonic flow over of a RAE2822 supercritical airfoil is presented with the aim of comparing the accuracy of the proposed methods. Such an assessment of the methods here implemented is achieved through the comparison of the present numerical results with data available in the literature.

2. Theoretical Formulation

In the present work, the 2-D Euler equations are solved in their integral form as

$$\frac{\partial}{\partial t} \int_V Q dV + \int_S (\vec{P} \cdot \vec{n}) dS = 0,$$

where $V$ represents the control volume, $S$ represents the surface of the control volume and $\vec{n}$ is the unit normal vector to the surface, $S$, positive outward. $Q$ is
the vector of conserved variables, and \( \vec{P} \) represents the sum of the convective flux vectors. The system is closed by the equation of state for a perfect gas.

The control volumes can be composed by any type of polygon, because the really important aspect is that its bounding contour can be decomposed into a finite number of line segments \( \Gamma_j \). The surface integral from Eq. (2.1) can be discretized using \( N \)-point Gaussian integration formulae

\[
\int_{S_i} (\vec{P} \cdot \vec{n})dS \approx \sum_j |\Gamma_j| \sum_{\ell=1}^N w_\ell \vec{P}(Q(G_\ell,t) \cdot \vec{n}), \tag{2.2}
\]

where \( G_\ell \) and \( w_\ell \) are, respectively, the Gaussian points and the weights on the \( \Gamma_j \) line segment. For the second-order schemes, one Gaussian point is necessary along each line segment. Its value is given by the edge midpoint and the respective weight, \( w_1 \), is chosen as \( w_1 = 1 \).

Using the method described above, one can compute values of \( Q_i \) in some instant, \( t \), and then, from these mean values, one can reconstruct polynomials that represent the primitive variables \( \rho, u, v \) and \( p \). Finally, it is possible to compute values of the conserved variables in the Gaussian points. Due to the discontinuity of the reconstructed values of the conserved variables over the cell boundaries, one must use a numerical flux function to approximate these flux values on the cell boundaries. In this work, the authors have used the Roe flux difference splitting method [16] to compute such approximations. A fully explicit second-order accurate TVD Runge-Kutta scheme was used to advance the solution of the governing equations in time, for the spatially second-order schemes, and this TVD Runge-Kutta scheme can be found in Shu and Osher [17].

3. ENO and WENO Reconstruction

The reconstruction procedure of the ENO schemes is based on the approximation of mean values of the primitive variables for each cell in the mesh by polynomials of one order less than the spatial order of accuracy expected. For the construction of polynomials of \( \eta-th \) order, one must use \( N(\eta) \) cells, where \( N(\eta) = (\eta + 1)(\eta + 2)/2 \).

The first step in obtaining the polynomial reconstruction for each cell is to define the possible set of cells, called stencil, that will be used. In the finite volume cell centered scheme the stencils can be selected in a von Neumann neighborhood for a linear polynomial reconstruction and can be extended to higher orders through the use of von Neumann neighborhoods of the primary neighbors already selected for the second-order reconstruction. The \( p(x, y) \) polynomials can, then, be calculated as

\[
p(x, y) = \sum_{|\beta| \leq \eta} r_{\beta_1, \beta_2} (x - x_c)^{\beta_1} (y - y_c)^{\beta_2},
\]

where \( |\beta| = \beta_1 + \beta_2 \), with \( \beta_i \in \{0, 1, 2, \ldots\} \), \( x_c \) and \( y_c \) are the Cartesian coordinates of the barycenter of the control volume and \( r_{\beta_1, \beta_2} \) are unknown coefficients which are some approximations to the derivatives of the primitive variables.
Once it is established that \( p(x, y) \) is a good approximation to the mean values of primitive variables for each cell one can write a linear system, \([R]\{r\} = \{\bar{u}\}\), of \( N(\eta) \) equations for the \( N(\eta) \) unknowns \( r_{\beta_1 \beta_2} \). Here, \([R]\) is the matrix of control volume moments as in Gooch [7], computed using the scaling technique proposed by Friedrich [6] to circumvent a poorly conditioned matrix. Moreover, \( \{r\} \) is the vector of unknown coefficients that must be found and \( \{\bar{u}\} \) is the vector with the mean values for each primitive variable. The stencil is considered admissible if \([R]\) matrix is invertible. The control volume moments that compose the \([R]\) matrix are defined by Eq. (3.1)

\[
\bar{x}^n\bar{y}^m \equiv \frac{1}{V_i} \int_{V_i} (x - x_c)^n(y - y_c)^m dV,
\]

and are evaluated using a Gauss quadrature formulae following the same procedure as the one used in the flux computing.

For the implementation of the second-order accurate schemes one must use three cells for the polynomial reconstructions. Hence, two unknown coefficients are computed for each polynomial through the solution of a two equation linear system composed by the control volume moments and the mean values of the primitive variables of the cells considered in the stencil. As only a maximum of three polynomials can be reconstructed by the use of three neighbors for a typical second-order reconstruction, there may be problems due to overshoots and undershoots with such a reconstruction. The algorithm implemented considers the ideas found in Sonar [18], where the neighborhood is extended. Therefore, the stencils are computed considering one primary neighbor and one secondary neighbor of the main control volume. This algorithm did not present any oscillation in the solutions and kept the schemes stable. In Figure 1 one can see a typical neighborhood for the implementation of the original ENO algorithm with a possible stencil (hatched volumes) for this algorithm and, in Figure 2, one can see a typical neighborhood for the implementation of the extended stencil algorithm with a possible stencil (hatched volumes).

The examples of neighborhoods presented here are for triangles and quadrilaterals. However, for the algorithms, it does not matter the geometry of the control volume. The control volumes can be formed by any type of polygon.

Figure 1: Neighborhood defined by the primary von Neumann neighbors of the main control volume. The hatched volumes exemplify a possible stencil for a typical second-order reconstruction.

After the polynomial reconstruction is performed for each cell, the next step is to verify which polynomial is the less oscillatory to use in the ENO scheme. The
oscillation is computed using some indicator that assesses the smoothness of \( p(x, y) \). Following the results presented in the literature [6], the oscillation indicator used in the present work is the one proposed by Jiang and Shu [10], which was later modified by Friedrich [6]. The formulation for this oscillation indicator can be expressed as

\[
OI_{JS}(p(x, y)) = \left[ \sum_{1 \leq |\beta| \leq \eta} \int_{V_i} h^{2|\beta|-4} \left( \frac{\partial |\beta| p(x, y)}{\partial x^{\beta_1} \partial y^{\beta_2}} \right)^2 \, dx \, dy \right]^{\frac{1}{2}}, \quad (3.2)
\]

where \( h \) is the mesh width.

Differently from the ENO schemes, the WENO schemes use all the calculated polynomials. These polynomials are added together through the use of weights which are computed for each one of the polynomials as proportional to its respective oscillation indicator. The main idea in the WENO reconstruction is to attribute the computed weights for each polynomial with the aim of reconstructing a new polynomial as \( p(x, y) = \sum_{k=1}^{m} \omega_k p_k(x, y) \). The weights are of order one in the smooth regions of the flow and are of order \( h^m \) in the regions with discontinuities, where \( h \) is the mesh width and \( m \) is the number of admissible stencils for the reconstruction of \( p_k(x, y) \). The weights can be computed as

\[
\omega_k = \frac{[\epsilon + OI(p_k(x, y))]^{-\theta}}{\sum_{k'=1}^{m} [\epsilon + OI(p_{k'}(x, y))]^{-\theta}},
\]

where \( \epsilon \) is a small real number used to avoid division by zero and \( \theta \) is a positive integer. The WENO schemes have the property of being very smooth and stable in smooth regions of the flow, but this property is lost if \( \theta \) is chosen too large. In that case the scheme tends to behave like the ENO schemes. In the present work, the \( \theta \) term is chosen as \( \theta = 2 \) because this yielded the best convergence rates in the results.
4. Multigrid Scheme

In this section, a general discussion of the implementation of a full approximation
storage (FAS) multigrid algorithm [20] is presented. With the use of the multigrid
procedure, the discretized problem for the fine mesh is solved in an approximate
manner on coarser meshes. As the coarse meshes have fewer points than the ini-
tial fine mesh, the solution of the problem in the coarse meshes requires a lower
computational cost than in the initial fine mesh. The properties of the fine mesh
are transferred to the coarse meshes by restriction operators and subsequently, the
solution in the fine mesh is updated by prolongation operators, which transfer the
properties from the coarse meshes. The restriction operator used in the present
work is the volume weighted average [14, 19], where the restricted conserved prop-
erties for the fine mesh cells forming the coarse mesh volumes are equal to the sum
of the conserved properties of all the fine mesh cells that compose the coarse mesh
volumes weighted by their volumes. The prolongation operator implemented in
the present work uses direct injection of the coarse mesh values into the fine mesh
[14, 19, 22].

The multigrid scheme implemented in the present work is based upon an agglom-
eration procedure [14, 19, 22] for generating the coarse meshes. A “seed” volume is
chosen in the fine mesh and, then, all the volumes that have a node or an edge in
common with this “seed” volume are grouped and they form the coarse mesh vol-
ume. Another “seed” volume is selected and the agglomeration procedure continues
grouping all the fine mesh volumes. It should be noted that, during the agglomera-
tion procedure, only the volumes that have not been already agglomerated may be
grouped to form a coarse mesh volume. This is a necessary condition in order to
guarantee that there is no volume overlapping in the coarse mesh. The selection of
the “seed” volumes is not random in order to obtain a better coarse mesh quality.
A list containing all the fine mesh volumes is generated prior to the agglomeration
procedure and the volumes close to the boundaries are chosen to be first selected
for the agglomeration. Further discussion of the agglomeration multigrid procedure
here used can be found in [20].

5. Results

The present section discusses computational results for the second-order schemes,
which were formulated for meshes composed exclusively by triangles. The objective
of the present study is to verify the newly implemented capability and to assess
its advantages/disadvantages with regard to the discretization methods previously
available in the code. Hence, the test case here analyzed was selected among those
for which well document, independent data are available in the literature.

The test case considered here is the transonic flow over a RAE2822 airfoil with
2.31 deg angle-of-attack. The freestream has a Mach number value of $M_{\infty} = 0.729$.
Density is made dimensionless with respect to the freestream condition and pressure
is made dimensionless with respect to the density times the speed of sound squared.
This case is computed using a mesh with 5435 nodes and 10260 triangular control
volumes. Two additional agglomerated grids are used in the computations for this
case and one can see a detail of the fine mesh and of the agglomerated multigrid meshes in Figure 3. This is a steady case and, hence, the CFL number is set at a constant value. The numerical results in terms of pressure contour lines are plotted in Figure 4 for the second-order WENO scheme using the triangular grid shown in Figure 3. One can observe the presence of the shock wave formed in the upper side of the airfoil.

![Meshes over the RAE2822](image1)

(a) Mesh over the RAE2822. (b) Mesh after the first agglomeration pass. (c) Mesh after the second agglomeration pass.

Figure 3: Detail of the meshes used in the simulation of the transonic flow over a RAE2822.

![Pressure contours](image2)

Figure 4: Pressure contours lines over the RAE2822 airfoil with 2.31 deg angle-of-attack for freestream Mach number $M_{\infty} = 0.729$.

The $C_p$ distributions along the airfoil chord are plotted in Figure 5. The results are obtained using the second-order Jameson scheme already validated in the code [3, 20], the second-order ENO and WENO schemes, the experimental data obtained from [15], and the numerical solution obtained with the MSES software [5]. The spatially second-order methods used the second-order TVD Runge-Kutta scheme for the temporal discretization. The ENO and WENO schemes considered Roe's method in the spatial discretization. Despite the fact that the present computations
are for inviscid flow, the results are in good agreement with the experimental data presented. One can see that, in the experimental data, the shock wave moves upstream over the airfoil due to the boundary layer presence. In the trailing edge of the airfoil the viscous effects are again visible in the solution.

Figure 5: Cp distribution along the chord of the RAE2822 airfoil with 2.31 deg angle-of-attack for freestream Mach number $M_{\infty} = 0.729$.

One can observe the better resolution of the WENO method, if compared to the others second-order methods, in capturing the shock over the airfoil. The centered scheme presents strong overshoots upstream and downstream the shock. The ENO scheme did not achieve the same Cp plateau upstream the shock as the other schemes and this scheme presented a dissipative behavior in the post-shock region. The second-order WENO scheme captured the shock with a smaller number of cells inside the shock, i.e., it captured the shock with a sharper definition. Although the numerical results obtained with the MSES code presented a lower Cp plateau in the region upstream the shock, the lower surface of the airfoil and the trailing edge region are in agreement with the numerical results here compared.

6. Conclusions

The reconstruction of essentially non-oscillatory (ENO) schemes and weighted essentially non-oscillatory (WENO) schemes is presented in this work. An approach for stencil selection are discussed for second-order ENO and WENO schemes. Although, in this paper, only second-order accurate schemes are actually implemented and assessed, the formulation of ENO and WENO reconstruction is treated in a
Compressible Aerodynamic Flow Simulations

A generic framework that allows the construction of polynomials of any order and, hence, of schemes with an arbitrary order of accuracy. The flux difference splitting scheme of Roe are used for flux evaluation in the edges, and a TVD Runge-Kutta time marching scheme is implemented for time advance of the 2-D Euler equations. A multigrid method is used for faster convergence to steady state solutions.

The test case studied in this work includes the transonic flow over the RAE2822 supercritical airfoil. This test case has its results compared with data available in the literature. The non-oscillatory schemes here implemented are compared with a well-known centered scheme, with a commercial code and with experimental data. The results confirm the more accurate solution of the WENO schemes comparing to the ENO schemes using the same stencils. The non-oscillatory schemes did not present any oscillation in the solutions contrary to the centered scheme.

Acknowledgments
The authors gratefully acknowledge the support of Fundação de Amparo à Pesquisa do Estado de São Paulo FAPESP through a Masters Scholarship for the first author under the grant 03/10047-2.

References


